ON SOLVABILITY OF THE VARIATIONAL INEQUALITY WITH +–COERCIVE MULTIVALUED MAPPINGS

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In this work the theory of variational inequalities with multivalued operators is extended for the wider of multivalued maps from the reflexive Banach space to its dual one. In particular, we relax the restriction on boundness of operators and on same properties of convergence. This operator class contains the bounded pseudomonotone maps ([6–8]), the maximal monotone maps on interior of domain ([2,5]), the s–weakly locally bounded generalized pseudomonotone maps ([3,10]) and other.

Let X be a reflexive Banach space, X^* be its topological dual space, $\langle\cdot,\cdot\rangle : X\times X^* \to \mathbb{R}$ be the duality pairing, $A : X \to 2^{X^*}$ be a multivalued mapping ($2^{X^*} = \bigcup\limits_{V\subset X^*} V$). We define $\mathrm{Dom}(A) = \{y \in X : A(y) \neq \varnothing\}$. We associated with A the lower and upper support functions $[A(y),\xi]_- = \inf\limits_{d\in A(y)} \langle d,\xi\rangle$ and $[A(y),\xi]_+ = \sup\limits_{d\in A(y)} \langle d,\xi\rangle$, where $y,\xi \in X$, $\|A(y)\|_+ = \sup\limits_{d\in A(y)} \|d\|_{X^*}$, $\|A(y)\|_- = \inf\limits_{d\in A(y)} \|d\|_{X^*}$. If $y \notin \mathrm{Dom}(A)$ then $[A(y),\xi]_- = +\infty$, $[A(y),\xi]_+ = -\infty$ for each $\xi \in X$ and $\|A(y)\|_- = \|A(y)\|_+ = 0$, $-\infty + \infty = +\infty$.

Definition 1. The mapping $A : \mathrm{Dom}(A) \subset X \to 2^{X^*}$ is **s–weakly locally bounded**, if for arbitrary $y \in \overline{\mathrm{Dom}(A)}$ and $\{y_n\} \subset \mathrm{Dom}(A)$, where $y_n \to y$ weakly in X, there exist the subsequence $\{y_{n_k}\}$ and $N > 0$ such that $\|A(y_{n_k})\|_+ \le N$.

Definition 2. The mapping $A : \mathrm{Dom}(A) \to 2^{X^*}$ **has the property** $(\mathfrak{M})$, if for arbitrary $\{(y_n,w_n)\} \subset \mathrm{graph}\,\overline{\mathrm{co}}\,A$, such that $y_n \to y$ weakly in X, $w_n \to w$ weakly in X^* and $\varlimsup\limits_{n\to\infty} \langle w_n, y_n - y\rangle \le 0$, we have that $w \in \overline{\mathrm{co}}A(y)$.

Definition 3. The mapping $A : \mathrm{Dom}(A) \to 2^{X^*}$ is **monotone**, if

$$[A(y_1), y_1 - y_2]_- \ge [A(y_2), y_1 - y_2]_+ \quad \forall y_i \in \mathrm{Dom}(A).$$

A is **maximal monotone** if it is monotone and for each monotone operator C $(\mathrm{graph}\,A \subset \mathrm{graph}\,C) \Rightarrow (C = A)$.

Definition 4. The mapping $A : \mathrm{Dom}(A) \to 2^{X^*}$ is **+–coercive on** K if there exist $y_0 \in K$ such that

$$\|y\|_X^{-1}[A(y), y - y_0]_+ \to +\infty \quad \text{as } \|y\|_X \to +\infty,\ y \in K.$$

This research is based on properties of multivalued maps which are perturbed by maximal monotone operators. In particular, we use the properties of maximal monotone operator ([2,3,5]), of operators with the properties $(\mathfrak{M})$ ([6–8] and its References) and of normal cone as the maximal monotone operator ([4]).

PROPOSITION. *Let* $A : \mathrm{Dom}(A) \subset X \to 2^{X^*}$ *is s–weakly locally bounded and has the property* $(\mathfrak{M})$*,* $B : \mathrm{Dom}(B) \to 2^{X^*}$ *is maximal monotone, then* $A + B$ *has the property* $(\mathfrak{M})$ *on* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$*.*

Proof. Let $y_n \to y$ weakly in X, $(A+B)(y_n) \ni w_n \to w$ weakly in X^* and $\overline{\lim\limits_{n\to\infty}} \langle w_n, y_n - y\rangle \le 0$. Since operator A is s–weakly locally bounded, there exists the subsequence $\{y_m\} \subset \{y_n\}$ such that $\|A(y_m)\|_+ \le N$, i.e. there exist $\{d_m \in \overline{\mathrm{co}}\, A(y_m)\}$ such that $\|d_m\|_{X^*} \le N$ and $\hat{w}_m = w_m - d_m \in B(y_m)$. Without restricting the generality, we can assume that $d_m \to d$ weakly in X^*. Thus, $B(y_m) \ni \hat{w}_m \to \hat{w}$ weakly in X^* and $\hat{w} + d = w$. Then or $\overline{\lim\limits_{m\to\infty}} \langle \hat{w}_m, y_m - y\rangle \le 0$ or $\overline{\lim\limits_{m\to\infty}} \langle d_m, y_m - y\rangle \le 0$ or both estimate hold. In first case $\hat{w} \in B(y)$ and $\langle \hat{w}_m, y_m\rangle \to \langle \hat{w}, y\rangle$ (see [2]). Thus, $\overline{\lim\limits_{m\to\infty}} \langle d_m, y_m - y\rangle = \overline{\lim\limits_{m\to\infty}} \langle w_m - \hat{w}_m, y_m - y\rangle = \overline{\lim\limits_{m\to\infty}} \langle w_m, y_m - y\rangle \le 0$. I.e. $d \in A(y)$, $w \in (A + B)(y)$. If $\overline{\lim\limits_{n\to\infty}} \langle d_n, y_n - y\rangle \le 0$ and $\overline{\lim\limits_{n\to\infty}} \langle \hat{w}_n, y_n - y\rangle > 0$, then $d \in A(y)$ by property $(\mathfrak{M})$ and $\exists \{y_m\}$ such that $\langle \hat{w}_m, y_m - y\rangle > \langle \hat{w}, y_m - y\rangle$. But B is maximal monotone, i.e. $\hat{w} \in B(y)$. ■

For proofing of based result we need the auxiliary property of multivalued operator on some closed convex set $D_r \subset B_r = \{y \in X : \|y\|_X \le r\}$, where $0 \in D_r$, ∂D_r is the boundary of D_r. Without restricting the generality the following property will be proved for convex– closed–valued operators.

THEOREM 1. *Let* $A : D_r \to 2^{X^*}$ *be a convex– closed–valued s–weakly locally bounded operator which has the property* $(\mathfrak{M})$*,* $A = \overline{\mathrm{co}} A$*,* $B : D_r \to 2^{X^*}$ *be maximal monotone and* $[(A+B)(y), y]_+ \ge 0$ *for each* $y \in \partial D_r$*. Then the inclusion* $0 \in \overline{\mathrm{co}}(A+B)(\xi)$*,* $\xi \in D_r$ *has nonempty weakly compact set of solutions.*

Sketch. Let $F(X)$ be a totality of finite-dimensional subspaces $F \subset X$. For arbitrary $F \in F(X)$ we introduce the projector $I_F : F \to X$ ($\|I_F y_F\|_X = \|y_F\|_F \, \forall y_F \in F$), $I_F^* : X^* \to F^*$ is a dual operator, $D_{rF} = D_r \cap F$, $A_F = A_{|F} : F \to 2^{X^*}$. And we introduce the auxiliaries operators $\forall y \in D_{rF} \equiv D_r \cap F$

$$I_F^* A_F(y) = \bigcup_{d \in A_F} \left\{ \sum_{\{h_i\}} \langle d(y), h_i\rangle h_i \right\}$$

$$I_F^* B_F(y) = \bigcup_{w \in B_F} \left\{ \sum_{\{h_i\}} \langle w(y), h_i\rangle h_i \right\}$$

$$I_F^* (A+B)_F(y) = \bigcup_{d \in A_F, w \in B_F} \left\{ \sum_{\{h_i\}} \langle w(y) + d(y), h_i\rangle h_i \right\}$$

where $\{h_i\}$ is the basis of F.

Using the properties of maximal monotone operators (see [2,5]) and s–weakly locally bounded operator with the properties $(\mathfrak{M})$ (see [7,8,10]), we can prove that the operator $I_F^*(A + B)_F$ is closed. Then by Remark 6.4.2 ([1]) for each $F \in F(X)$ there exists $y_F \in D_{rF}$ such that $0 \in I_F^*(A + B)_F(y_F)$. We can construct the system with the finite intersection property $\{\overline{G}_F^w\}$, where $\overline{G}_F^w$ is the weak closure of $G_{F_0} = \bigcup\limits_{F \supset F_0} \Big\{ y_F \in D_{rF} : 0 \in I_F^*(A + B)_F(y_F) \Big\}$. Since X is reflexive, then $\exists\, y \in \bigcap\limits_{F \in F(X)} \{\overline{G}_F^w\}$. And by property $(\mathfrak{M})$ $0 \in (A + B)(y)$. ■

For the bounded demiclosed operator with the property α) this theorem had been considered in [9].

THEOREM 2. *The variational inequality (VIMO)*

$$[A(y), \xi - y]_+ \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K$$

has at least one bounded solution if the mapping A is $+$-coercive, s–weakly locally bounded and A satisfies the property ($\mathfrak{M}$) on K. Moreover, there exists the solution of VIMO y which is bounded by constant R ($\|y\|_X \leq R$) where $[A(\xi), \xi - y_0]_+ \geq 0$ for each $\xi \in K$ such that $\|\xi\|_X = R$.

Sketch. It suffices to show that the inclusion $\overline{\text{co}}A(y) + \partial I_K(y) \ni f$, where I_K is the characteristic map of K ($I_K(y) = 0$ as $y \in K$ and $I_K(y) = +\infty$ as $y \notin K$). the subdifferentional ∂I_K is normal cone (see [4]), i.e. it is maximal monotone (see [2]). By the definition $\partial I_K(y) = \{w \in X^* : \langle w, \xi\rangle \leq 0 \quad \forall \xi \in \overline{\frac{1}{h}\bigcup\limits_{h>0}(K - y)}\}$, thus, $0 \in \partial I_K(\xi)$ for each $\xi \in K$ and $[\partial I_K(y), y - y_0]_+ = +\infty$ as $y \in \partial K$ (∂K is the boundary of K). Since $[A(y) - f + \partial I_K(y), y - y_0]_+ = [A(y) - f, y - y_0]_+ + [\partial I_K(y), y - y_0]_+$, at least one of summed tends to infinity as $\|y\|_X \to \infty$, i.e. there exists the constant $R > 0$ such that $[A(y) + \partial I_K(y), y - y_0]_+ \geq 0$ if $y \in \partial(K \cap B_R(y_0))$. Thus, by Theorem 1 there exists $\hat{y}$ such that $0 \in \overline{\text{co}}A(\hat{y}) + \partial I_K(\hat{y})$.

Let $\{y_n\}$ be subset of solution set, $y_n \to y$ weakly in X, $0 \in \overline{\text{co}}A(y_n) + \partial I_K(y_n)$. Then $\overline{\lim}\langle 0, y_n - y\rangle = 0$. By the property ($\mathfrak{M}$) $0 \in \overline{\text{co}}A(y) + \partial I_K(y)$, consequently, the set of solution is nonempty and weakly compact. ■

COLLORARY. *Let A be s–weakly locally bounded and satisfies the property ($\mathfrak{M}$) on K, K be closed convex and bounded. Then VIMO has nonempty weakly compact set of solutions.*

Example. Let Ω be a bounded set from $\mathbb{R}^n$, Γ be the boundary of Ω. We consider such Ω that Γ is smooth. And we study the free-boundary problem on Sobolev space $W^1_p(\Omega)$, where $p \geq 2$, $p^{-1} + q^{-1} = 1$, the track $y_{|\Gamma}$ belongs to $W^{1/q}_p(\Gamma)$. Let $\nu(x)$ be the normal in $x \in \Gamma$. Let us consider the free-boundary problem

$$A(y) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(a_i(x,y)\frac{\partial y}{\partial x_i}\big) + a_0(x,y)y = f,$$

$$y_{|\Gamma} \geq 0, \qquad \frac{\partial y}{\partial \nu_A} + \gamma(y) \cap [0, \infty) \neq \varnothing, \qquad y\big(\frac{\partial y}{\partial \nu_A} + \gamma(y)\big) \ni 0,$$

where $\frac{\partial y}{\partial \nu_A} = \sum\limits_{i=1}^{n} a_i(x,y)\frac{\partial y}{\partial x_i}\cos(\nu_i, x)$, $\gamma : W^{1/q}_p(\Gamma) \to 2^{W^{-1/q}_q(\Gamma)}$ is a s-weakly locally bounded operator which has the property ($\mathfrak{M}$). And let a_i satisfy to Caratheodori conditions and following ones:

$$|a_i(x,y)| \leq g(x) + C_{1i}|y|^{p-1},$$

where $g \in L_{q'}(\Omega)$, $q' = \frac{p}{p-1}$, if $p > 2$, and $g \in C^\infty(\Omega)$, if $p = 2$;

$$a_i(x,y) > C_{2i}|y|^{p-2}, \qquad \text{for sufficiently large } |y| >> R\,(C_{2i} > 0).$$

Then we can consider the integral forms, i.e. we can construct the variational inequality

$$\sum_{i=1}^{n}\int_{\Omega} a_i(x,y)\frac{\partial y}{\partial x_i}\frac{\partial(\xi-y)}{\partial x_i}dx+\int_{\Omega} a_0(x,y)y(\xi-y)dx+$$
$$+[\gamma(y),\xi-y]_+\geq\int_{\Omega} f(\xi-y)dx \quad \forall\xi\in K,$$

where $[\gamma(y),\xi-y]_+=\sup_{v\in\gamma(y)}\int_{\Gamma} v(\xi-y)d\Gamma$, $K=\{y\in W_p^1(\Omega): y_{|\Gamma}\geq 0\}$. By Theorem 2 this VIMO has at least one solution for each $f\in W_q^{-1}(\Omega)$, this statement holds if some values of γ are not convex and closed. Also this statement holds if for some (not all) selector $d\in\gamma$ $d(y)\to-\infty$ as $\|y\|_X\to\infty$.

In [2,8] the problem had been considered for the operator with $a_i(x,y)\equiv a_i(x)$ and maximal monotone operator γ. In [10] this problem had been considered for generalized pseudomonotone operator γ, the proving of the properties is analogous.

References

1. Aubin J.–P., Ekland I., *Applied Nonlinear Analysis*, In: J.Wiley and Sons., Inc. (1984).
2. Barbu V., *Analysis and control of nonlinear infinite dimensional systems*, In: Acad. Press, Inc. (1995).
3. Browder F.E., Hess P., *Nonlinear Mappings of Monotone Type in Banach Spaces*, J. Func. Anal. **11** (1972), no. 2, pp. 251-294.
4. Clarce F., *Optimization and Nonsmooth analysis*, In: J.Wiley and Sons., Inc. (1983).
5. Z.Guan and A.G.Kartsatos., *Ranges of Perturbed Maximal Monotone and m-Accretive Operators in Banach Spaces*, Trans.of AMS **347** (1995), 2403–2435.
6. Ivanenko V.I. and Mel'nik V.S., *Variational Methods in Control Problems for Distributed Systems*, In: Kiev: Naukova dumka (in Russian) (1988).
7. Mel'nik V.S., Solonoukha O.V., *On the Stationary Variational Inequalities with the Multivalued Operators*, Cybernetics and System Analysis **3** (1997), 74–89(in Russian).
8. Mel'nik V.S., Solonoukha O.V., *On the Variational Inequalities with the Multivalued Operators*, Dokl.NAN of Ukraine **5** (1997), 33–39(in Russian).
9. Mel'nik V.S., Vakulenko A.N., *On the same Class of Operator Inclusions in Banach Spaces*, Dokl.NAN of Ukraine (1997), (in print) (in Russian).
10. Solonoukha O.V., *On the Stationary Variational Inequalities with the Generalized Pseudomonotone Operators*, Methods of Functional Analysis and Topology (1997), (in print).

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