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STABILITY OF INVARIANT SETS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. Systems of functional differential equations with delay $dz(t)/dt = Z(t, z_t)$ and $dz(t)/dt = Z(t, z_t) + R(t, z_t)$ are considered where $z = (x, y), x \in \mathbb{R}^n, y \in \mathbb{R}^m$, and Z and R are the vector-valued functionals. It is supposed that these systems have a positive invariant set $x = 0$. The conditions are given when the uniform asymptotic stability of the invariant set of the first system implies the uniform asymptotic stability of the invariant set of the second system. The asymptotic stability of this invariant set of the first system is studied separatly when the right-hand side of the system is an almost periodic in t.

1. Introduction

Let $t \in \mathbb{R}_+ = [0, \infty)$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, $|x| = \sqrt{\sum_{i=1}^n (x^i)^2}$, $y = (y^1, \dots, y^m)$, $|y| = \sqrt{\sum_{s=1}^m (y^s)^2}$, $z = (x, y) = (z^1, \dots, z^{n+m}) \in \mathbb{R}^{n+m}$, $|z| = \sqrt{|x|^2 + |y|^2}$. For a given $h > 0$, C^n and C^m denote the spaces of continuous functions mapping $[-h, 0]$ into $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^{n+m}) = (\psi, \lambda)$, where $\psi = (\psi^1, \dots, \psi^n) \in C^n$, $\lambda = (\lambda^1, \dots, \lambda^m) \in C^m, C = C^n \times C^m$. Denote

$$\|\psi\| = \sup(|\psi^i(\theta)|, \quad \text{under} \quad -h \le \theta \le 0, 1 \le i \le n),$$
$$\|\lambda\| = \sup(|\lambda^j(\theta)|, \quad \text{under} \quad -h \le \theta \le 0, 1 \le j \le m),$$
$$\|\varphi\| = \max(\|\psi\|, \|\lambda\|),$$
$$C_H = \{\varphi \in C : \|\psi\| \le H, \|\lambda\| < +\infty\}.$$

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If z is a continuous function of u defined on $-h \leq u < A, A > 0$, and if t is a fixed number satisfying $0 \leq t < A$, then z_t denotes the restriction of z to the segment $[t-h, t]$ so that $z_t = (z_t^1, \dots, z_t^{n+m}) = (x_t, y_t)$ is an element of C defined by $z_t(\theta) = z(t+\theta)$ for $-h \leq \theta \leq 0$.

Consider a system of functional differential equations

$$\frac{dz(t)}{dt} = Z(t, z_t). \tag{1.1}$$

In this system dz/dt denotes the right-hand derivative of z at t, t is time, and $Z(t,\varphi) = (X(t,\varphi), Y(t,\varphi)) \in \mathbb{R}^{n+m}$ is defined on $\mathbb{R}_+ \times C_{H_1}$; $X \in \mathbb{R}^n, Y \in \mathbb{R}^m$. Such systems were studied in [1,2,5-11,14-17,19-23,26-32].

According to T.Burton [5], we denote by $z(t_0,\varphi) = (x(t_0,\varphi), y(t_0,\varphi))$ a solution of (1.1) with initial condition $\varphi \in C_{H_1}$, where $z_{t_0}(t_0,\varphi) = \varphi$ and we denote by $z(t, t_0, \varphi)$ the value of $z(t_0,\varphi)$ at t and $z_t(t_0,\varphi) = z(t+\theta, t_0, \varphi)$, $-h \leq \theta \leq 0$.

It is assumed that the vector-valued functional $Z(t,\varphi)$ is continuous on $\mathbb{R}_+ \times C_{H_1}$ so that a solution will exist for each continuous initial condition. We suppose that each solution $z(t_0,\varphi)$ is defined for those $t \geq t_0$, such that $\|x_t(t_0,\varphi)\| < H_1$.

Let $V(t,\varphi)$ be a continuous functional defined for $t \geq 0, \varphi \in C_{H_1}$. The upper right-hand derivative of V along solutions of (1.1) is defined to be [5,19,26]

$$\begin{aligned}\dot{V}(t, z_t(t_0,\varphi)) &= \frac{dV(t, z_t(t_0,\varphi))}{dt} \\ &= \overline{\lim_{\Delta t \to +0}} \{V(t+\Delta t, z_{t+\Delta t}(t_0,\varphi)) - V(t, z_t(t_0,\varphi))\} \frac{1}{\Delta t}.\end{aligned}$$

If V satisfies a Lipschitz condition in the second argument, then this limit is finite.

In [12,24,25,33-40] the partial stability results were obtained for ordinary differential equations, and in [10,27,39,40] the partial stability results were obtained for functional differential equations with delay. Consider the set

$$M := \{\varphi \in C : \|\psi\| = 0, \|\lambda\| < \infty\}. \tag{1.2}$$

The necessary and sufficient conditions of the uniform asymptotic stability of the invariant set M of system (1.1) were obtained in [4]. In that paper, the method of Lyapunov functionals, founded by N. Krasovskii [28], was used. It was proved there that for uniform asymptotic stability of M it is necessary and sufficient the existence of continuous functional $V : \mathbb{R}_+ \times C_H \to \mathbb{R} \quad (H < H_1)$

such that

$$a(\|x_t\|) \le V(t, z_t) \le b(\|x_t\|), \quad a, b \in \mathcal{K}, \tag{1.3}$$

$$\frac{dV}{dt} \le -c(\|x_t\|), \quad c \in \mathcal{K} \tag{1.4}$$

along solutions of system (1.1). Here $\mathcal{K}$ denotes the class of Hahn's functions [18,35], that is $r \in \mathcal{K}$ if $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonically increasing function such that $r(0) = 0$. Note that in [6,7,23] these functions are called *wedges*.

The purpose of this paper is twofold. First we consider the system

$$\frac{dz(t)}{dt} = Z(t, z_t) + R(t, z_t) \tag{1.5}$$

for which M is also the invariant set. In section 3 the restrictions on R are stated under which the uniform asymptotic stability of M of system (1.1) implies the uniform asymptotic stability of invariant set (1.2) of system (1.5). In section 4 we also consider the particular case of system (1.1) when Z is an almost periodic function of t. It is shown that for asymptotic stability of M of system (1.1) it is sufficiently the existence of a functional V which has more weak properties than (1.3) and (1.4).

2. The basic definitions and notations

We shall consider the set M of form (1.2).

Definition 2.1. A set $M \subset C$ is called a positive invariant set of system (1.1) if $t_0 \in \mathbb{R}_+$, $\varphi \in M$ implies $z_t(t_0, \varphi) \in M$ for each $t \ge t_0$.

In the next definitions it is assumed that M is a positive invariant set of system (1.1).

Definition 2.2. A set $M \subset C$ is called a stable set of system (1.1) if for every $\varepsilon > 0$ and $t_0 \ge 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C$, $\|\psi\| < \delta$ implies $\|x_t(t_0, \varphi))\| < \varepsilon$ for all $t \ge t_0$.

Definition 2.3. If δ does not depend on t_0 in Definition 2.2 (i.e., $\delta = \delta(\varepsilon)$), then the set M is called uniformly stable.

Definition 2.4. A set $M \subset C$ is called an attractive set of system (1.1) if for every $t_0 \ge 0$ there exists $\eta = \eta(t_0) > 0$, and for every $\varepsilon > 0$ and $\varphi \in C$, $\|\psi\| < \eta$ there exists $\sigma = \sigma(\varepsilon, t_0, \varphi) > 0$ such that $\|x_t(t_0, \varphi)\| < \varepsilon$ for any $t \ge t_0 + \sigma$. In this case we say that the domain of attraction of M at t_0 contains the set C_η. In other words, a set M is called attractive if

$$\lim_{t\to\infty} \|x_t(t_0,\varphi)\| = 0. \tag{2.1}$$

Definition 2.5. A set $M \subset C$ is called uniformly attractive set of system (1.1) if for some $\eta > 0$ and any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that $\|x_t(t_0,\varphi)\| < \varepsilon$ for all $t_0 \geq 0$, $\varphi \in C_\eta$ and $t \geq t_0 + \sigma$. In other words, a set M is called uniformly attractive if (2.1) holds uniformly in $t_0 \in \mathbb{R}_+, \varphi \in C_\eta$.

Definition 2.6. A set M is called:
-asymptotically stable if it is stable and attractive;
-uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

3. On the uniform asymptotic stability of M of perturbed systems

Along with equations (1.1) consider the system

$$\frac{dw(t)}{dt} = Z(t,w_t) + R(t,w_t), \tag{3.1}$$

where $w(t) = (u(t),v(t)), u \in \mathbb{R}^n, v \in \mathbb{R}^m$, $w_t = (u_t,v_t) = (w_t^1,...,w_t^{n+m})$; $u_t = (u_t^1,...,u_t^n)$, $v_t = (v_t^1,...,v_t^m)$. Suppose that functionals $Z(t,\varphi)$ and $R(t,\varphi)$ are uniformly bounded in the domain $\mathbb{R}_+ \times C_{H_1}$ and satisfy the conditions

$$\begin{aligned} |Z_i(t,\varphi_1) - Z_i(t,\varphi_2)| &\leq L^*\|\psi_1 - \psi_2\|, \\ |R_i(t,\varphi_1) - R_i(t,\varphi_2)| &\leq L^*\|\psi_1 - \psi_2\|, \end{aligned} \tag{3.2}$$

where $\varphi_1 = (\psi_1,\lambda_1), \varphi_2 = (\psi_1,\lambda_2)$, L^* =const, $1 \leq i \leq n$. We also assume that

$$Z_i(t,\varphi) = 0, \quad R_i(t,\varphi) = 0 \quad \text{under} \quad \varphi_i = 0 \quad (i = 1,...,n),$$

so equations (3.1) have the solution w such that

$$u(t) = 0, \tag{3.3}$$

and M is a positive invariant set of system (3.1).

Lemma 3.1. *Let $v : \mathbb{R}_+ \times C_H \to \mathbb{R}$ $(H < H_1)$ be a continuous functional satisfying*

$$|v(t,\varphi_1) - v(t,\varphi_2)| \leq L\|\psi_1 - \psi_2\|, \tag{3.4}$$

where $\varphi_1 = (\psi_1,\lambda_1)$, $\varphi_2 = (\psi_2,\lambda_2)$, L =const. Then the inequality

$$\left.\frac{dv(t,w_t(t_0,\varphi))}{dt}\right|_{t=t_0} \leq \frac{dv(t,z_t(t_0,\varphi))}{dt} + L \sup_{\substack{\varphi\in C_H \\ 1\leq i\leq n}} |R_i(t_0,\varphi)|$$

holds in the domain $\mathbb{R}_+ \times C_H$.

Proof. From the definition of the derivative of the functional along solutions of the system of functional differential equations with delay, we have

$$\begin{aligned}
&\frac{dw(t, w_t(t_0, \varphi))}{dt}\bigg|_{t=t_0} \\
&= \lim_{\Delta t \to +0} \sup \frac{1}{\Delta t}[v(t_0 + \Delta t, w_{t_0+\Delta t}(t_0, \varphi)) - v(t_0, w_{t_0}(t_0, \varphi))] \\
&\leq \lim_{\Delta t \to +0} \sup \frac{1}{\Delta t}[v(t_0 + \Delta t, z_{t_0+\Delta t}(t_0, \varphi)) - v(t_0, z_{t_0}(t_0, \varphi))] \\
&\quad + \lim_{\Delta t \to +0} \sup \frac{1}{\Delta t}[v(t_0 + \Delta t, w_{t_0+\Delta t}(t_0, \varphi)) - v(t_0 + \Delta t, z_{t_0+\Delta t}(t_0, \varphi))] \\
&\leq \frac{dv(t, z_t(t_0, \varphi))}{dt}\bigg|_{t=t_0} + L \lim_{\Delta t \to +0} \frac{1}{\Delta t} \sup \|u_{t_0+\Delta t}(t_0, \varphi) - x_{t_0+\Delta t}(t_0, \varphi)\| \\
&\leq \frac{dv(t, z_t(t_0, \varphi))}{dt}\bigg|_{t=t_0} + L \sup_{\substack{\varphi \in C_H \\ 1 \leq i \leq n}} |R_i(t_0, \varphi)|.
\end{aligned}$$

This completes the proof. □

Definition 3.1. We shall say that a functional $Q : \mathbb{R}_+ \times C_H \to \mathbb{R}^{n+m}$ satisfies condition (B_1) if there is a $\beta > 0 (\beta < H)$ such that for any $\xi \in (0, \beta)$ there exist a $\tau_\xi \geq 0$ and a function $g_\xi(t)$, continuous on $[\tau_\xi, \infty)$ such that $|Q_i(t, \varphi)| \leq g_\xi(t)$ $(i = 1, \dots, n)$ for $\varphi \in C_\beta \setminus C_\xi, t \in [\tau_\xi, \infty)$, and

$$\lim_{t \to \infty} G_\xi(t) = 0 \tag{3.5}$$

where $G_\xi(t) = \int_t^{t+1} g_\xi(s)ds$.

Definition 3.2. We shall say that a functional $Q : \mathbb{R}_+ \times C_H \to \mathbb{R}^{n+m}$ satisfies condition (B_2) if there is a $\beta > 0$ $(\beta < H)$ such that $|Q_i(t, \varphi)| \leq g(t)(i = 1, \dots, n)$ for $\varphi \in C_\beta, t \in \mathbb{R}_+$, and

$$\lim_{t \to \infty} G(t) = 0, \quad \text{where} \quad G(t) = \int_t^{t+1} g(s)ds.$$

It is clear that, if a functional Q satisfies condition (B_2), then it satisfies condition (B_1).

Henceforth we shall suppose that M is a positive invariant set for systems (1.1) and (3.1).

Theorem 3.1. *Let M be a uniformly asymptotic stable set of system* (1.1)*, and its domain of attraction contains C_H. If $R(t, \varphi)$ satisfies condition (B_1), then M is also a uniformly asymptotic stable set of system* (3.1)*, and there* *exists a positive* $\eta(\eta < H)$ *such that the domain of attraction of* M *of system* (3.1) *contains* C_η.

Proof. According to [4], there exists a functional $v : \mathbb{R}_+ \times C_H \to \mathbb{R}$ satisfying inequalities (3.4) and

$$a(\|\psi\|) \le v(t,\varphi) \le b(\|\psi\|), \quad a,b \in \mathcal{K}, \tag{3.6}$$

$$\frac{dv(t,z_t)}{dt} \le -c(\|x_t\|), \quad c \in \mathcal{K}.$$

Pick any $\varepsilon > 0(\varepsilon < \beta)$. Denote $\xi := b^{-1}(\frac{1}{2}a(\varepsilon))$. Then inequalities (3.6) imply

$$\inf_{\|\psi\|=\varepsilon} v(t,\varphi) \ge a(\varepsilon), \qquad \sup_{\|\psi\|\le\xi} v(t,\varphi) \le b(\xi) = \frac{1}{2}(\varepsilon). \tag{3.7}$$

Let us show that each trajectory $w(t_1,\overline{\varphi}) = w(t_1,\overline{\psi},\overline{\lambda})$ of equations (3.1) satisfies $\|u_t\| < \varepsilon$ for $t > t_1 \ge t_*(\varepsilon)$ where t_* is so large that

$$t_* > 1, \quad t_* > \tau_\xi, \tag{3.8}$$

and $\|\overline{\psi}\| = \xi$. Suppose not: there is a system of functional differential equations (3.1), satisfying the above conditions, which has a solution

$$w(t_1,\overline{\varphi}), \tag{3.9}$$

satisfying conditions $w_{t_1} = \overline{\varphi}$, $\|u_{t_1}\| = \xi$, $\|u_{t_2}\| = \varepsilon$. We also assume that trajectory (3.9) satisfies

$$\xi \le \|u_t\| \le \varepsilon \tag{3.10}$$

for $t \in [t_1,t_2]$. Using Lemma 3.1 and inequalities (3.10) we estimate the derivative of functional v along solution (3.9) of system (3.1):

$$\frac{dv(t,w_t(t_1,\overline{\varphi}))}{dt} \le -c(\xi) + L \sup_{\substack{\varphi\in C_\varepsilon \\ 1\le i\le n}} |R_i(t,\varphi)|. \tag{3.11}$$

From (3.11) we get

$$\begin{aligned}
\Delta v &= v(t_2, w_{t_2}(t_1,\overline{\varphi})) - v(t_1, w_{t_1}(t_1,\varphi)) \\
&\le -c(\xi)(t_2-t_1) + L\int_{t_1}^{t_2} \sup_{\substack{\varphi\in C_\varepsilon \\ 1\le i\le n}} |R_i(t,\varphi)|dt \\
&\le -c(\xi)(t_2-t_1) + L\int_{t_1}^{t_2} g(\xi)(s)ds \\
&\le -c(\xi)(t_2-t_1) + L\int_{t_1-1}^{t_2} G_\xi(s)ds.
\end{aligned}$$

Consider the function

$$E_\xi(t) = \sup_{s\in[t-1,\infty)} G_\xi(s)ds.$$

In view of (3.5), this function is monotonically nonincreasing and satisfies the condition

$$\lim_{t\to\infty} E_\xi(t) = 0.$$

From this limit relation we obtain

$$\Delta v \le -c(\xi)(t_2-t_1)+LE_\xi(t_*)(t_2-t_1+1) = (-c(\xi)+LE_\xi(t_*))(t_2-t_1)+LE_\xi(t_*).$$

Functionals Z_i and R_i $(i = 1,\ldots,n)$ are uniformly bounded in the set $\mathbb{R}_+ \times C_H$, hence there exists $\Delta T > 0$, depending only on ε, such that $t_2 - t_1 \ge \Delta T$. We shall assume that t_* is so large that

$$LE_\xi(t_*) < \frac{1}{2}c(\xi), \quad LE_\xi(t_*) < \frac{1}{4}c(\xi)\Delta T. \tag{3.12}$$

The values of ξ and ΔT in inequalities (3.8), (3.12) depend only on ε, therefore t_* may be chosen depending only on ε. Then $\Delta v \le -\frac{1}{4}c(\xi)(t_2 - t_1)$ in view of (3.8) and (3.12). This contradicts (3.7) and implies that there do not exist t_1, $t_2(t_2 > t_1 \ge t_*(\varepsilon))$ such that $\|u_{t_1}\| = \xi(\varepsilon)$, $\|u_{t_2}\| = \varepsilon$ for (3.1).

Each solution $w(t_0,\varphi)$ of functional differential equations (3.1) depends continuously on initial conditions. Consequently, in view of inequality (3.2) of [4] and (3.2), there exists a $\delta > 0$ such that for all $t_0 \in [0,t_*]$, $\varphi \in C_\delta$ we have $\|u_{t_*}(t_0,\varphi)\| \le \xi$. Since ξ and t_* depend only on ε, then δ also depends only on ε. This proves that M is uniformly stable for (3.1). Now let us show that M is uniformly asymptotically stable. Let q be any fixed positive number $(q < H)$. We have proved there exists $\eta(q) > 0$ such that each trajectory $w(t_0,\varphi)$ of system (3.1) satisfying the initial condition $\|u_{t_0}\| < \eta$, satisfies inequality $\|u_t\| < q$ for all $t > t_0 \ge 0$. Let us show that for every $\rho > 0$ $(\rho < q)$ there is a $\sigma = \sigma(\rho) > 0$ such that the inequality $\|u_t(t_0,\varphi)\| < \rho$ holds for arbitrary $t_0 \in \mathbb{R}_+$, $\varphi \in C_\eta$, $t \ge t_0 + \sigma$.

Let $0 < \rho < q$; we have proved there exists $\delta = \delta(\rho) > 0$ such that $u_{T_0} \in C_\delta$ implies $u_t \in C_\rho$ for every $t > T_0 \ge 0$. Let us estimate the time for which the element of the trajectory w_t will lie in the domain $C_q \setminus C_\delta$.

Similarly to the above, one can show that

$$\Delta V = V(t,w_t) - V(T_1,w_{T_1}) \le -\frac{1}{4}c(\delta)(t - T_1) \tag{3.13}$$

for $t \ge T_1$, where T_1 depends only on $\delta(\rho)$, i.e. $T_1 = T_1(\rho)$. Then for $t \ge T_1$ inequality (3.13) implies

$$t - T_1 \le \frac{4[V(T_1,w_{T_1}) - V(t,w_t)]}{c(\delta)} \le 4\frac{V(T_1,w_{T_1})}{c(\delta)} \le 4\frac{b(\delta(\rho))}{c(\delta(\rho))} = T_2(\rho).$$

Setting $\sigma(\rho) = T_1(\rho) + T_2(\rho)$, we obtain that the inequality $\|u_t(t_0, \varphi)\| < \rho$ is valid for all $t_0 \in \mathbb{R}_+$, $\varphi \in C_\eta$, $t \geq t_0 + \sigma$. Hence the positive invariant set M of system (3.1) is uniformly asymptotically stable, and its domain of attraction contains C_η. This completes the proof of Theorem 3.1. □

Side by side with system (1.1), consider the next system of functional differential equations with delay

$$\frac{dw(t)}{dt} = Z(t, w_t) + R(t, w_t) + Q(t, w_t), \tag{3.14}$$

where $w(t) = (u(t), v(t))$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m, w_t = (u_t, v_t)$, $u_t = (u_t^1, \dots, u_t^n)$, $v_t = (v_t^1, \dots, v_t^m)$. Suppose that functionals $Z(t, \varphi), R(t, \varphi)$ and $Q(t, \varphi)$ are uniformly bounded in $\mathbb{R}_+ \times C_{H_1}$ and satisfy the conditions

$$\begin{aligned} |Z_i(t, \varphi_1) - Z_i(t, \varphi_2)| &\leq L^* \|\psi_1 - \psi_2\|, \\ |R_i(t, \varphi_1) - R_i(t, \varphi_2)| &\leq L^* \|\psi_1 - \psi_2\|, \\ |Q_i(t, \varphi_1) - Q_i(t, \varphi_2)| &\leq L^* \|\psi_1 - \psi_2\|, \end{aligned}$$

where $\varphi_1 = (\psi_1, \lambda_1)$, $\varphi_2 = (\psi_2, \lambda_2)$, L^* =const, $1 \leq i \leq n$ and

$$Z_i(t, \varphi) = 0, R_i(t, \varphi) = 0, Q_i(t, \varphi) = 0 \quad \text{under} \quad \varphi_i = 0 (i = 1, \dots, n),$$

so equations (3.14) have the solution w such that (3.3) holds, and M is an invariant set of system (3.1).

Theorem 3.1 and [4] imply the following corollary.

Corollary 3.1. *Let M be a uniformly asymptotic stable set of system* (1.1), *and its domain of attraction contains C_H. If $R(t, \varphi)$ satisfies*

$$\lim_{t \to \infty} \int_t^{t+\tau} R_i(t, \varphi) dt = 0, \quad 1 \leq i \leq n$$

uniformly in $\tau > 0$, $\varphi \in C_H$, and $Q(t, \varphi)$ satisfies condition (B_1), *then M is also a uniformly asymptotic stable set of system* (3.14), *and there exists a positive $\eta (\eta < H)$ such that the domain of attraction of M of system* (3.14) *contains C_η.*

4. On the stability of the positive invariant set in almost periodic systems

Consider system (1.1), and suppose that $Z(t, \varphi)$ is bounded in the domain $\mathbb{R}_+ \times C_H$. We also assume that

$$Z_i(t, 0, \lambda) \equiv 0, \quad i = 1, \dots, n,$$

so M is an invariant set of system (1.1).

Definition 4.1. [13,26] A continuous function $F : \mathbb{R} \to \mathbb{R}^{n+m}$ is called almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha+l], \alpha \in \mathbb{R}$, contains at least one number τ such that $|F(t+\tau)-F(t)| < \varepsilon$ for every $t \in \mathbb{R}$. A number τ is called an ε-almost period of F.

Let $A > 0, B > 0$. We denote

$$C_{A,B} := \{\varphi \in C_A : \|\lambda\| \leq B\};$$
$$C_{A,B(L)} := \{\varphi \in C_{A,B} : |\varphi(\theta_1) - \varphi(\theta_2)| \leq L|\theta_1 - \theta_2| \text{ for each } \theta_1, \theta_2 \in [-h, 0]\}.$$

Definition 4.2. A continuous functional $F : \mathbb{R} \times C \to \mathbb{R}^{n+m}$ is called uniformly almost periodic in t with respect to $\varphi \in C_{A,B(L)}$ if for every $\varepsilon > 0$ there exists $l = l(\varepsilon, A, B, L) > 0$ such that any segment $[\alpha, \alpha + l]$, $\alpha \in \mathbb{R}$, contains at least one number τ such that $|F(t + \tau, \varphi) - F(t, \varphi)| < \varepsilon$ for all $t \in \mathbb{R}, \varphi \in C_{A,B(L)}$.

Lemma 4.1. [26] *Let $F_1(t), \ldots, F_N(t) : \mathbb{R} \to \mathbb{R}^{n+m}$ be almost periodic functions. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l]$, $\alpha \in \mathbb{R}$, contains a number τ such that*

$$|F_i(t + \tau) - F_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, N; t \in \mathbb{R}.$$

Lemma 4.2. *If the functional $F(t, \varphi) : \mathbb{R} \times C \to \mathbb{R}^{n+m}$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_{A,B(L)}$, then it is uniformly almost periodic in t with respect to $\varphi \in C_{A,B(L)}$.*

Proof. Since the functional $F(t, \varphi)$ satisfies Lipschitz conditions in φ, then

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq L_1\|\varphi_1 - \varphi_2\| \tag{4.1}$$

where L_1 is the Lipschitz constant.

Let $\varepsilon > 0$ be any real number. $C_{A,B(L)}$ is the set of uniformly bounded equicontinuous functions, therefore $C_{A,B(L)}$ is a compact set. Hence there is a finite set of functions $\varphi_1, ..., \varphi_N$ such that $\varphi_j \in C_{A,B(L)}$ $(j = 1, ..., N)$ and for each $\varphi \in C_{A,B(L)}$ there exists such number i $(1 \leq i \leq N)$ that

$$\|\varphi - \varphi_i\| < \frac{\varepsilon}{3L_1}. \tag{4.2}$$

From Lemma 4.1 it follows that there exists $l > 0$ such that in any segment $[\alpha, \alpha + l]$ there exists a number τ such that

$$|F(t, \varphi_i) - F(t + \tau, \varphi_i)| < \frac{\varepsilon}{3} \tag{4.3}$$

for each $t \in \mathbb{R}, i = 1, ..., N$.

Now we will show that for every $\varphi \in C_{A,B(L)}$, each number τ, which satisfies inequality (4.3), is an ε-almost period of the functional $F(t, \varphi)$. Let φ_k be the same element of the set $\varphi_1, ..., \varphi_N$ for which $\|\varphi - \varphi_k\| < \varepsilon/(3L_1)$. Then from (4.1)-(4.3) we obtain

$$\begin{aligned}|F(t+\tau,\varphi) - F(t,\varphi)| &\le |F(t+\tau,\varphi) - F(t+\tau,\varphi_k)| \\ &\quad + |F(t+\tau,\varphi_k) - F(t,\varphi_k)| + |F(t,\varphi_k) - F(t,\varphi)| \\ &< \frac{\varepsilon}{3} + 2L_1 \cdot \frac{\varepsilon}{3L_1} = \varepsilon \end{aligned} \tag{4.4}$$

The inequality (4.4) proves Lemma 4.2. □

We consider the system of functional differential equations under the assumptions above. We also assume that the functional $Z(t,\varphi)$ is bounded, Lipschitzian in φ, and almost periodic in t for every fixed $\varphi \in C_{A,B(L)}(0 < A < H, B > 0)$.

Lemma 4.3. [26] *Consider the solution $z(t_0,\varphi_0)$ of system* (1.1). *We suppose that $z_t(t_0,\varphi_0)$ belongs to $C_{A,B}$ $(0 < A < H, B > 0)$ for $t \ge 0$. Let $\{\varepsilon_k\}$ be a monotonically approaching zero sequence of positive numbers and $\{\tau_k\}$ a sequence of ε_k-almost periods of $Z(t,\varphi)$ (for every ε_k there corresponds an ε_k-almost period τ_k). Then*

$$\lim_{k\to\infty} \|z_{t^*}(t_0,\varphi_k) - z_{t^*+\tau_k}(t_0,\varphi_0)\| = 0 \tag{4.5}$$

holds, where $\varphi_k = z_{t_0+\tau_k}(t_0,\varphi_0)$ and t^* is a fixed moment of time which is more than $t_0 (t^* > t_0)$.

Definition 4.3. The solution $z(t_0,\varphi)$ of system (1.1) is called x-finally nonzero if for every $t > t_0$ there exists $t_* > t$ such that $|x(t_*,t_0,\varphi)| \ne 0$.

Theorem 4.1. *Let functional differential equations* (1.1) *satisfy the above conditions*; *let any solution $z(t_0,\varphi)$ such that $z_t(t_0,\varphi) \in C_H$ be y-bounded, and there exists a continuous functional $V(t,\varphi) : \mathbb{R} \times C_H \to \mathbb{R}$, which is locally Lipschitz in φ, such that the following conditions are fulfilled on the set $\mathbb{R} \times C_H$:*

(i) $V(t,0,\lambda) \equiv 0,\ a(|\psi(0)|) \le V(t,\varphi) \le b(\|\psi\|)$, *where $a, b \in \mathcal{K}$;*
(ii) *$V(t,\varphi)$ is almost periodic in t for each fixed $\varphi \in C_{A,B}(0 < A \le H, B > 0)$;*
(iii) *$dV/dt \le 0$, $dV/dt \not\equiv 0$ on each x-finally nonzero solution of system* (1.1).

Then M is asymptotically stable set of system (1.1).

Proof. Pick any $\varepsilon_1 > 0$ $(\varepsilon_1 < H)$, and choose $\delta = b^{-1}(a(\varepsilon_1))$, where b^{-1} is the inverse of the function b. If $\|\psi\| < \delta$, then from conditions (i) and (iii) we have

$$a(|x(t,t_0,\varphi)|) \le V(t,z_t) \le V(t_0,\varphi) \le b(\|\psi\|) < b(b^{-1}(a(\varepsilon_1))) = a(\varepsilon_1),$$

whence if follows $|x(t)| < \varepsilon_1$ for $t > t_0$. This proves the uniform stability of M.

Now we shall prove that M is attractive. Let $\varepsilon \in (0, H)$ be any positive number. Denote by $t_0 \in \mathbb{R}$ the initial moment of time. The uniform stability of M implies that there exists a $\delta > 0$ such that if $\varphi \in C_\delta$, then $z_t(t_0, \varphi) \in C_\varepsilon$ for every $t \geq t_0$. Choose such a $\delta > 0$ and show that $z_t(t_0, \varphi)$ with $\varphi \in C_\delta$ tends to M as t tends to infinity.

The function $V(t) = V(t, z_t(t_0, \varphi))$ is monotonically nonincreasing because $dV/dt \leq 0$. Hence there exists the limit

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V(t, z_t(t_0, \varphi)) = V_0,$$

and it is easy to see that $V(t, z_t(t_0, \varphi)) \geq V_0 \geq 0$ for $t \in [t_0, \infty)$. Let us show that $V_0 = 0$. Suppose that this is not true; i.e. assume that $V_0 > 0$.

The solution $z(t_0, \varphi)$ is y-bounded, and $z_t(t_0, \varphi) \in C_\varepsilon$. Hence this solution is bounded, i.e. $z_t(t_0, \varphi) \in C_{A,B}$ for some positive A, B. Therefore there is a $L > 0$ such that $|Z(t, z_t(t_0, \varphi))| < L$ for $t > t_0 + h$. This means that $z_t(t_0, \varphi) \in C_{A,B(L)}$ for $t > t_0 + h$. Consider some monotonically approaching zero sequence $\{\varepsilon_k\}$ of positive numbers, where ε_1 is sufficiently small. By Lemma 4.2, for every ε_i there exists a sequence of ε_i-almost periods $\tau_{i,1}, \tau_{i,2}, ..., \tau_{i,n}, ... \to \infty$ for functionals $Z(t, \varphi)$ and $V(t, \varphi)$ that inequalities

$$|V(t + \tau_{i,n}, \varphi) - V(t, \varphi)| < \varepsilon_i, \quad |Z(t + \tau_{i,n}, \varphi) - Z(t, \varphi)| < \varepsilon_i$$

hold for each $t \in \mathbb{R}, \varphi \in C_{A,B(L)}$. Without loss of generality we suppose $\tau_{i,n} < \tau_{i+1,n}$ for every i, n. Designate $\tau_k = \tau_{k,k}$.

Consider the sequence of functions $\varphi_k = z_{t_0+\tau_k}(t_0, \varphi)$ $(k = 1, 2, ...)$. It is a bounded sequence of equicontinuous functions; therefore there is a limit function φ^* of this sequence. Without loss of generality we can assume the sequence φ_k itself converges to φ^*. Because of continuity and almost periodicity of the functional $V(t, \varphi)$ we obtain

$$\begin{aligned} V(t_0, \varphi^*) &= \lim_{n\to\infty} V(t_0, \varphi_n) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} V(t_0 + \tau_k, \varphi_n) = \lim_{n\to\infty} V(t_0 + \tau_n, \varphi_n) \\ &= \lim_{n\to\infty} V(t_0 + \tau_n, z_{t_0+\tau_n}(t_0, \varphi)) = V_0 \end{aligned}$$

Now consider the solution $z(t_0, \varphi^*)$. From condition (iii) of the theorem, the existence of such moment of time $t^*(t^* > t_0)$ follows when the inequality

$$V(t^*, z_{t^*}(t_0, \varphi^*)) = V_1 < V_0$$

holds.

Solutions of functional differential equations (1.1) are continuous in initial data, so one can write

$$\lim_{k\to\infty} \|z_{t^*}(t_0,\varphi_k) - z_{t^*}(t_0,\varphi^*)\| = 0$$

because

$$\lim_{k\to\infty} \|\varphi_k - \varphi^*\| = 0.$$

Hence it follows

$$\lim_{k\to\infty} V(t^*, z_{t^*}(t_0,\varphi_k)) = V_1. \tag{4.6}$$

Using the uniform almost periodicity property of $Z(t,\varphi)$ and limit relation (4.5), we obtain the inequality

$$\|z_{t^*}(t_0,\varphi_k) - z_{t^*+\tau_k}(t_0,\varphi)\| \le \gamma_k, \tag{4.7}$$

where $\gamma_k \to 0$ as $k \to \infty$. Because of uniform almost periodicity property of $V(t,\varphi)$ we have

$$|V(t^*,\varphi) - V(t^* + \tau_k,\varphi)| < \varepsilon_k \tag{4.8}$$

for every $\varphi \in C_H$ and from conditions (4.6) and (4.7) it follows that

$$|V(t^*, z_{t^*+\tau_k}(t_0,\varphi)) - V_1| < \eta_k, \tag{4.9}$$

where $\eta_k \to 0$ as $k \to \infty$.

From (4.8) we obtain

$$|V(t^*, z_{t^*+\tau_k}(t_0,\varphi)) - V(t^* + \tau_k, z_{t^*+\tau_k}(t_0,\varphi))| < \varepsilon_k. \tag{4.10}$$

From (4.9),(4.10) we have

$$|V(t^* + \tau_k, z_{t^*+\tau_k}(t_0,\varphi)) - V_1| < \eta_k + \varepsilon_k, \tag{4.11}$$

where $\eta_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand

$$\lim_{k\to\infty} V(t^* + \tau_k, z_{t^*+\tau_k}(t_0,\varphi)) = V_0. \tag{4.12}$$

The relations (4.11), (4.12) are in contradiction to the inequality $V_1 < V_0$. The obtained contradiction proves that $V_0 = 0$. This completes the proof. □

References

1. Z. Artstein and M. Slemrod, *On Singularly Perturbed Retarded Functional Differential Equations*, J. of Differential Equations **171** (2001), no. 1, 88-109.
2. Z. S. Athanassov, *Families of Liapunov-Krasovskii functionals and stability for functional differential equations*, Ann. Mat. Pure Appl. **176** (1999), no. 4, 145-165.

3. A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*; Lecture Notes in Control and Information Sciences 267, Springer Verlag, London, 2001.
4. S. Bernfeld, C. Corduneanu, and A. O. Ignatyev, *On the stabitity of invariant sets of functional differential equations*, Nonlinear Analysis (to appear).
5. T. A. Burton, *Uniform Asymptotic Stability in Functional Differential Equations*, Proceedings of the American Mathematical Society **68** (1978), no. 2, 195-199.
6. T. A. Burton and L. Hatvani, *Stability theorems for nonautonomous functional-differen -tial equations by Liapunov fuctionals*, Tøhoku Math. J. **41** (1989), no. 2, 65-104.
7. T. A. Burton and L. Hatvani, *On nonuniform asymptotic stability for nonautonomous functional-differential equations*, Differential Integral Equations **3** (1974), 285-293.
8. T. A. Burton and G. Makay, *Asymptotic stability for functional-differential equations*, Acta Math. Hungar **65** (1994), 243-251.
9. C. Corduneanu, *Some differential equations with delay*, Proceedings of EQUADIFF 3, Brno, 1973, pp. 105-114.
10. C. Corduneanu, *On partial stability for delay systems*, Ann. Polon. Math. **29** (1975), 357-362.
11. C. Corduneanu, *Functional Equations with Causal Operators*, Taylor and Francis, London and New York, 2002.
12. C. Corduneanu, *Some problems concerning partial stability*, Symp. Math. Meccanica nonlineare. Stability. 23-26 febbrario, 1970, vol. 6, Acad. Press, New York, 1971, pp. 141-154.
13. C. Corduneanu, *Almost Periodic Functions*, 2nd edition, Chelsea Publ. Co., New York, 1989.
14. C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay*, Avtomatika i Telemekhanika (1985), no. 7, 5-44. (Russian)
15. C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey*, Nonlinear Analysis: TMA **4** (1980), no. 5, 831-877.
16. A. Domoshnitsky, M. Drakhlin, E. Litsyn, *On equations with delay depending on solutions*, Nonlinear Analysis **49** (2002), no. 5, 689-701.
17. I. V. Gaishun and L. B. Knyazhishche, *Nonmonotone Lyapunov functionals. Conditions for the stability of equations with delay*, Differential Equations **30** (1994), 1195-1200.
18. W. Hahn, *Stability of Motion*, Springer, New York-Berlin-Heidelberg, 1967.
19. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York-Heidel -berg-Berlin, 1977.
20. I. Győri and F. Hartung, *Stability in Delay Perturbed Differential and Difference Equations*, Fields Institute Communications **29** (2001), 181-194.
21. I. Győri, F. Hartung, and J. Turi, *Preservation of Stability in Delay Equations under Delay Perturbations*, J. Math. Anal. Appl. **220** (1998), 290-312.
22. L. Hatvani, *On the asymptotic stability for functional differential equations by Lyapunov functionals*, Nonlinear Analysis **47** (2001), no. 7, 4333-4343.
23. L. Hatvani, *On the asymptotic stability for nonautonomous functional differential equations by Lyapunov functionals*, Transactions of the American Math. Society **354** (2002), no. 9, 3555-3571.
24. A. O. Ignatyev, *Partial stability under persistent disturbances*, Izv. Vyssh. Uchebn. Zaved. Mat. (1991), no. 2(345), 55-60. (Russian)
25. A. O. Ignat'ev, *Preservation of the property of uniform asymptotic stability with respect to some of the variables*, J. Appl. Math. Mech. (1989), no. 53, 136-139.

26. A. O. Ignatyev, *On the Asymptotic Stability in Functional Differential Equations*, Proceedings of the American Mathematical Society **127** (1999), no. 6, 1753-1760.
27. M. A. Kalistratova, *On the partial stability of the systems with delay*, Avtimatika i Telemekhanika **47** (1986), 32-37. (Russian)
28. N. N. Krasovskii, *Stability of Motion*, Stanford University Press, Stanford, California, 1963.
29. V. Lakshmikantham, *Uniform asymptotic stability criteria for functional differential equations in terms of two measures*, Nonlinear Analysis **34** (1998), no. 1, 1-6.
30. V. Lakshmikantham, S. Leela, and S. Sivasundaram, *Liapunov functions on product spaces and stability theory of delay differential equations*, J. Math. Anal. Appl. **154** (1991), 391-402.
31. X. Liu, *Stability in terms of of two measures for functional differential equations*, J. Diff. Integr. Equat. (1989), no. 3, 257-261.
32. X. Liu and D. Y. Xu, *Uniform asymptotic stability of abstract functional-differential equations*, J. Math. Anal. Appl. **216** (1997), 626-643.
33. K. Peiffer and N. Rouche, *Liapunov's second method applied to partial stability*, J. Mecanique **6** (1969), no. 2, 20-29.
34. C. Risito, *Sulla stabilita asintotica parziale*, Ann. Math. Pura Appl. **84** (1970), 279-292.
35. N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York, 1977.
36. V. V. Rumyantsev and A. S. Oziraner, *Partial Stability and Stabilization of Motion*, Nauka, Moscow, 1987. (Russian)
37. A. Ya. Savchenko and A. O. Ignatyev, *Some Problems of Stability Theory*, Naukova Dumka, Kiev, 1989. (Russian)
38. E. D. Sontag and Y. Wang, *Lyapunov characterizations of input to output stability*, SIAM J. Control and Optim. **39** (2001), no. 1, 226-249.
39. V. I. Vorotnikov, *Partial Stability and Control*, Birkhauser, Boston, 1998.
40. V. I. Vorotnikov and V. V. Rumyantsev, *Stability and control with respect to a part of the phase coordinats of dynamic systems: theory, methods, and applications*, Scientific World, Moscow, 2001. (Russian)

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