BOUNDARY VALUE PROBLEM FOR CERTAIN CLASSES OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH FREE BOUNDARY

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In the interval $(0, t_0)$ the following problem will be considered

$$m(t)\frac{d\nu}{dt} = k_1 F(t) - f_1(\nu), \tag{1}$$

$$-k_2 \frac{dm}{dt} = F(t), \tag{2}$$

$$f_2(\nu) + K_3 F(t) = m(t) K_n, \tag{3}$$

$$m(0) = M_0,\ V(0) = V_0, \tag{4}$$

$$m(t_0) = m_0, \tag{5}$$

where $f_1(V)$ and $f_2(V)$ are given continuous on V functions at $V \geq 0$, k_1, k_2, k_3, k_4, M_0, m_0 and V are given positive constants, $m(t), F(t)$ and $V(t)$ unknown functions to be found in the interval $(0, t_0)$, satisfying the conditions

$$0 < m_0 < M_0,\ f_1(0) = 0,\ f_2(V_0) < k_4 M_0, \tag{6}$$

$$f_1'(V) > 0,\ f_2'(V) > 0 \quad \text{at} \quad V > 0, \tag{7}$$

$$f_1'(V) \to +\infty, f_2'(V) \to +\infty \quad \text{at} \quad V \to +\infty.$$

In this section we prove the following theorems.

Theorem 1. The problem (1) - (5) is uniquely solvable. To prove theorems 1 and 2 we need the following lemma.

Lemma 1. Let $F(t)$, $V(t)$ and $m(t)$ be the solution of the problem (1) - (4) in the interval $0 < t < t_0$ and $m(t)$ satisfy the condition

$$m_0 \leq m(t) \leq M_0 \tag{8}$$

then $V(t) > 0, F(t) > 0$ at $0 \leq t < t, t_0 < \infty$.

Proof of lemma 1. Let $V(t)$ be equal to zero at some point $\tau, 0 < \tau < t_0$ and τ be the smallest number such that $V(\tau) = 0$. As $V(0) = V_0 > 0$ then $V(t) > 0$ at $0 \leq t < \tau$, so

$$V'(\tau) \leq 0. \tag{9}$$

Substituting into (3) $t = \tau$ and recalling the equality $f_1(0) = 0$ and inequality (8), we get

$$F(\tau) \geq \frac{m_0 k_4}{k_3}. \tag{10}$$

Putting into (1) $t = \tau$, we get

$$m(\tau)V'(\tau) = F(\tau)k_1. \tag{11}$$

From (9), (10) and (11) we have

$$V'(\tau) > 0. \tag{12}$$

The condition (12) is a contradiction with (9). This means that the supposition $V(\tau) = 0$ at some point $\tau \in (0, t_0)$ is not true. Hence

$$V(\tau) > 0 \quad \text{at} \quad 0 \leq t < t_0.$$

We prove now that $F(t) > 0$ at $0 \leq t < t_0$. From the condition $f_2(V_0) < M_0 k_4$ and equality (3) at $t = 0$ it follows that $F(0) > 0$. Let $F(t)$ be equal to zero at some point of the interval $(0, t_0)$. Then, as for (9), we can show that there exists a point $\tau \in (0, t_0)$ where

$$F(\tau) = 0, \quad F'(\tau) \leq 0. \tag{13}$$

Putting in (2) $t = \tau$, we get

$$m'(\tau) = 0. \tag{14}$$

Differentating the both sides of (3) with respect to t and substituting $t = \tau$, we get

$$V'(\tau)f_2'(V(\tau)) + F'(\tau)k_3 = 0. \tag{15}$$

As $V(\tau) > 0$, the condition (7) implies $f_2'(V(\tau)) > 0$. So from (13) and (15) we have

$$V'(\tau) \geq 0. \tag{16}$$

Putting into (1) $t = \tau$ and recalling the condition $F(\tau) = 0$ we obtain

$$m(\tau)V'(\tau) = -f_1(V(\tau)). \tag{17}$$

As $f_1(0) = 0$ and $f'(V) > 0$ at $V > 0$, then $f_1(V) > 0$ at $V > 0$. So from (17) it follows that

$$V'(\tau) < 0. \tag{18}$$

The condition (18) is a contradiction with (16). This means that $F(t)$ is not equal to zero in the interval $(0, t_0)$. As $F(0) > 0$ we get $F(t) > 0$ at $0 \leq t < t_0$.

We prove now that $t_0 < \infty$. Integrating the inequality (2) from zero to t, $t \in (0, t_0)$ and using the inequality (8) we have

$$\int_0^{t_0} F(t)dt \leq k_2(M_0 - m_0). \tag{19}$$

As $F(t) > 0$ the inequality (19) implies that the left - hand side has a limit at $t \to t_0$. Passing in (19) to the limit at $t \to t_0$, we get

$$\int_0^{t_0} F(t)dt \le k_2(M_0 - m_0). \tag{20}$$

Let ω and Ω be two subsets of the interval $(0, t_0)$ satisfying the conditions

$$F(t) \le \frac{m_0 k_4}{2k_3} \quad \text{at} \quad t \in \omega, \tag{21}$$

$$F(t) \le \frac{m_0 k_4}{2k_3} \quad \text{at} \quad t \in \Omega, \tag{22}$$

respectively. Denote by ω_0 and Ω_0 the Lebesgue measure of the sets ω and Ω,

$$0 \le \omega_0 \le t_0, \Omega_0 = t_0 - \omega_0.$$

It is clear that

$$\int_0^{t_0} F(t)dt \ge \int F(t)dt \ge \omega_0 \frac{m_0 k_4}{2k_3}. \tag{23}$$

From (20) and (23) follows the inequality

$$\omega_0 \le \frac{2k_2 k_3 (M_0 - m_0)}{k_4 m_0}. \tag{24}$$

From the inequalities (10), (22) and the equality (3) we have

$$f_2(V(t)) \ge \frac{m_0 k_4}{2} \tag{25}$$

Dividing the both sides of (1) by $m(t)$ we obtain

$$\frac{f_1(V(t))}{m(t)} = \frac{k_1 F(t)}{m(t)} - V'(t). \tag{26}$$

Integrating (26) from zero to $t, (t \in (0, t_0))$ we get

$$\int_0^t \frac{f_1(V(t))}{m(t)} dt = k_1 \int_0^t \frac{F(t)}{m(t)} dt - V'(t) + V. \tag{27}$$

As $V(t) > 0$ and $F(t) > 0$ at $t \in (0, t_0)$ and $m_0 \le m(t) \le M_0$, (29) implies

$$\int_0^t \frac{f_1(V(\tau))d\tau}{m(\tau)} \le \frac{k_1}{m_0} \int_0^t F(\tau)d\tau + V_0. \tag{28}$$

The last inequality and (20) yield

$$\int\limits_0^t \frac{f_1(V(\tau))d\tau}{m(\tau)} \leq \frac{k_1k_2}{m_0}(M_0 - m_0) + V_0, \quad t \in (0, t_0). \tag{29}$$

As $V(\tau) > 0, f_1(V(\tau)) > 0$ and $m(\tau) > 0$, from the inequality (29) we deduce that the left - hand side has a limit at $t \to t_0$ and this limit satisfies the inequality

$$\int\limits_0^t \frac{f_1(V(\tau))d\tau}{m(\tau)} \leq V_0 + \frac{k_1k_2}{m_0}(M_0 - m_0). \tag{30}$$

Let c_0 be a positive solution of the equation

$$f_2(c_0) = \frac{m_0k_4}{2}. \tag{31}$$

As $f_1(V)$ and $f_2(V)$ are increasing functions, from (25) and (31) it follows that

$$V(t) \geq c_0 \quad \text{at} \quad t \in \Omega, \tag{32}$$

$$f_1(V(t)) \geq f_1(c_0) \quad \text{at} \quad t \in \Omega. \tag{33}$$

From (10) and (33) we have

$$\int\limits_0^{t_0} \frac{f_1(V(\tau))d\tau}{m(\tau)} \leq \int\limits_\Omega \frac{f_1(V(\tau))d\tau}{m(\tau)} \geq \frac{f_1(c_0)}{M_0}\Omega_0. \tag{34}$$

From the inequalities (30) and (34) we obtain

$$\Omega_0 \leq \frac{M_0}{f_1(c_0)}\left[V_0 + \frac{k_1k_2}{m_0}(M_0 - m_0)\right]. \tag{35}$$

From the relations (4) and (35) we get $t_0 < \infty$. Lemma 1 is proved.

Proof of theorem 1. It is known that the problem (1) - (4) possesses a solution in a sufficiently small neighbourhood $(0, \varepsilon)$ (. [1], [2]).

As $F(0) > 0$, from (2) deduce that $m(t)$ is a monotone decreasing in this neighbourhood function an

$$m_0 < m(t) < M_0 \quad \text{at} \quad t \in (0, \varepsilon). \tag{36}$$

Let $(0, t_0)$ be the maximal neighbourhood where the solution of the problem (1) - (4), satisfying the inequality

$$m_0 < m(t) < M_0 \quad \text{at} \quad t \in (0, t_0) \tag{37}$$

exists.

According to lemma 1, the interval $(0, t_0)$ is bounded.

As $m(t)$ is decreasing in interval $(0, t_0)$ and satisfies the inequality (37) the limit of $m(t)$ at $t \to t_0$ exists. Denote this limit by $m(t_0)$. It is clear that $m_0 < m(t) < M_0$. We show that $m(t_0) = m_0$.

According to lemma 1 $V(t) > 0$ and $F(t) > 0$ at $t \in (0, t_0)$. Hence $f_2(V)$ is also positive. Thus from (3) and (37) we get

$$f_2(V) \le M_0 k_4, \quad k_3 F(t) \le M_0 k_4. \tag{38}$$

So

$$0 \le V \le c_1, \quad F(t) \le \frac{M_0 k_4}{k_3} \quad \text{at} \quad 0 \le t < t_0, \tag{39}$$

where c_1 is a positive solution of the equation $f_2(c_1) = M_0 k_4$. The inequalities (37) and (39) imply that the right - hand side of (1) is bounded in the interval $(0, t_0)$. This means that is also bounded in this interval. Hence there exists the limit of $V(t)$ at $t \to t_0$. Denote this limit by $V(t_0)$. Therefore, from (1) - (3) we may conclude that $F(t), m(t)$ and $V(t)$ are continously derivable on the segment $[0, t_0]$ functions. As in the case of lemma 1, we can show that

$$F(t_0) > 0, V(t_0) > 0. \tag{40}$$

Let $m(t_0) \ne m_0$. Then

$$m_0 < m(t_0) < M_0. \tag{41}$$

Denote $V(t_0) = V_1$, $m(t_0) = m_1$. Consider Cauchy problem for the system of equations (1) - (3) in the interval $(t_0, t_0 + \varepsilon)$ with boundary conditions

$$V(t_0) = V_1, m(t_0) = m_1. \tag{42}$$

As it is know this problem admits a solution for sufficiently small ε. The double inequality $m_0 < m(t_0) < M_0$ shows that letting ε to be small one might ensure the inequality $m_0 < m(t_0) < M_0$ at $t_0 \le t \le t_0 + \varepsilon$. So the solution of the problem (1)-(4), satisfying the inequality (37) in the interval $(0, t_0 + \varepsilon)$ exists. But this is not possible as $(0, t_0)$ is the maximal neighbourhood, where such a solution exists. This means that our supposition, namely $m(t_0) \ne m_0$, is not true, hence $m(t_0) = m_0$. Theorem 1 is proved.

The above results are used in mathematical modelling of the flight of winged aircraft along a given trajectory.

References

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