GENERALIZED SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS WITH STRONG POWER SINGULARITIES

© H.P.Lopushanska

The existence of a generalized solution of elliptic boundary value problems and its character of singularities depending on power of data singularities are established.

In [1-3] and other articles the behavior of the generalized solutions to the elliptic boundary value problems with power singularities on the right-side was studied. It was established that the generalized solutions in the sense [4] exist if the power growth of the problem data has the order $\lambda > -n$ inside the domain and $\lambda > max\{1-n, -n+2m-1-m'\}$ on its boundary (m' denotes the maximum order of the normal derivatives in the boundary conditions) and that these data require a regularization if their power growth is more strong [1,2].

We trace the behavior of generalized solutions (in specific sense) for any power singularities on the right-side without any using of the data regularization. We start from a representation of the solution.

Consider the following problem

$$A(x, D) = F_0, x \in \Omega, B_j(x, D)u \mid_S = F_j, \quad j = \overline{1, m}, \tag{1}$$

where Ω denotes a bounded domain in $\mathbb{R}^n$, with a closed boundary S of class C^∞, $A(x, D)$ is an elliptic operator of order $2m$, $\{B_j(x, D)\}_{j=1}^m$ are some normal system of boundary differential expressions satisfying Lopatinsky's condition. We assume that the coefficients of operators are infinitely differentiable.

Let $\hat{B}_j, T_j, \hat{T}_j$ be such boundary differential operators with the infinitely differentiable coefficients that Green's formula

$$\int_\Omega (Auv - uA^*v)dx = \sum_{j=1}^m \int_S (B_j u\hat{T}_j v - T_j u\hat{B}_j v)dS$$

holds.

We define now the following function spaces: $D(\overline{\Omega}) = C^\infty(\overline{\Omega})$, $D(S) = C^\infty(S)$, $X(\overline{\Omega}) = \{\varphi \in D(S) : \hat{B}_j\varphi \mid_S = 0, j = \overline{1, m}\}$ and $D'(\overline{\Omega}), D'(S), X'(\overline{\Omega})$ as spaces of linear continious functionals defined respectively on $D(\overline{\Omega}), D(S), X(\overline{\Omega})$. Let (φ, F) denote the action of $F \in D'(\overline{\Omega})(X'(\overline{\Omega}))$ onto $\varphi \in D(\overline{\Omega})(X(\overline{\Omega}))$ and $< \varphi, F >$–the action of $F \in D'(S)$ onto $\varphi \in D(S)$.

For the case $F_0 \in X'(\overline{\Omega}), F_j \in D'(S), j = \overline{1, m}$, we define the solution of the problem (1) as such generalized function $u \in D'(\overline{\Omega})$ that the equality

$$(A^*\psi, u) = (\psi, F_0) + \sum_{j=1}^m < \hat{T}_j\psi, F_j > \tag{2}$$

is fulfilled for any $\psi \in X(\overline{\Omega})$.

In [4,6] there is established the existence and is studied the properties of Green's vector-function $(G_0(x,y), G_1(x,y), \dots, G_m(x,y))$ of the problem (1) on the class of the functions $u(x)$ which are orthogonal to the kernel N of the problem (1) (i.e. $Pu = 0$).

If

$$(\psi, F_0) + \sum_{j=1}^{m} < \hat{T}_j \psi, F_j >= 0 \qquad (3)$$

for any $\psi \in N^*$ (N^* is the kernel of the adjoint problem), the solution of the problem (1) in the sense (2) exists in $D'(\overline{\Omega})/N$. It is defined by the formula

$$(\varphi, u) = (\int\limits_{\overline{\Omega}} \varphi(x) G_0(x,y) dx, F_0) + \sum_{j=1}^{m} < \int\limits_{\overline{\Omega}} \varphi(x) G_j(x,y), F_j >, \qquad (4)$$
$$\varphi \in D(\overline{\Omega}).$$

The function with strong power singularities doesn't belong to $D'(\overline{\Omega})$ or $X'(\overline{\Omega})$ but (4) can be extend to this case also. We consider special function spaces for it.

Let x_0 denote a given point in $\overline{\Omega}$, $\varrho(x, x_0) = \varrho_0(x - x_0)$ be nonnegative compactly supported infinitely differentiable function in $\overline{\Omega}$, wich has order $d(x, x_0) = |x - x_0|$ in neighborhood of the point $x_0 \in \overline{\Omega}$, $\varrho(x_0, x_0) = 0$.

For $k \in \mathbb{R}^1$ we define spaces $Z_k(\overline{\Omega}, x_0) = \{\varphi \in C^\infty(\overline{\Omega} \setminus x_0) : \varrho^{|\alpha|}(x, x_0) D^\alpha \varphi(x) = \varrho^k(x, x_0)\varphi_\alpha(x), \varphi_\alpha(x) \in C(\overline{\Omega})$ for arbitrary multi-index $\alpha\}$.

We shall say that the sequence $\varphi_\nu \to 0$ in the space $Z_k(\overline{\Omega}, x_0)$, if for all multi-index α the sequence $\varphi_{\alpha\nu}(x) = \varrho^{-k+|\alpha|}(x, x_0) D^\alpha \varphi_\nu(x)$ uniformly tends to zero in $\overline{\Omega}$ under $\nu \to \infty$.

We now notice the following main properties of the functions of these spaces:

1) $Z_0(\overline{\Omega}, x_0) \subset C(\overline{\Omega})$, $C^\infty(\overline{\Omega}) \subset Z_k(\overline{\Omega}, x_0)$ for all $k \leq 0$;

2) if $\varphi \in Z_k(\overline{\Omega}, x_0)$, than, for all $\lambda \in \mathbb{R}^1$, $|x - x_0|^\lambda \varphi \in Z_{k+\lambda}(\overline{\Omega}, x_0)$;

3) if $\varphi \in Z_k(\overline{\Omega}, x_0)$, than $D^\gamma \varphi \in Z_{k-|\gamma|}(\overline{\Omega}, x_0)$ for any multi-index γ;

4) if $\varphi \in Z_k(\overline{\Omega}, x_0)$, $\psi \in Z_p(\overline{\Omega}, x_0)$, than $\varphi\psi \in Z_{k+p}(\overline{\Omega}, x_0)$;

5) $Z_{k_2}(\overline{\Omega}, x_0) \subset Z_{k_1}(\overline{\Omega}, x_0)$ for $k_1 < k_2$;

6) $Z_k(\overline{\Omega}, x_0) \subset C^{[k]}(\overline{\Omega})$ for $k \geq 0$ (here $[k]$ denotes the integral part of the number k).

We denote the spaces of linear continuous functionals defined on $Z_k(\overline{\Omega}, x_0)$ by $Z'_k(\overline{\Omega}, x_0)$. Then

1) $Z'_{k_1}(\overline{\Omega}, x_0) \subset Z'_{k_2}(\overline{\Omega}, x_0)$ for $k_1 < k_2$; if $k \geq 0$, than $(C^{[k]}(\overline{\Omega}))' \subset Z'_k(\overline{\Omega}, x_0)$; since $C^\infty(\overline{\Omega}) = D(\overline{\Omega}) \subset Z_{-k}(\overline{\Omega}, x_0)$ for $k \geq 0$, $Z'_{-k}(\overline{\Omega}, x_0) \subset D'(\overline{\Omega})$ for $k \geq 0$.

2) If $F \in Z'_k(\overline{\Omega}, x_0)$, than $D^{|\alpha|}F \in Z'_{k+|\alpha|}(\overline{\Omega}, x_0)$ for any multi-index α.

3) $Z_{-k}(\overline{\Omega}, x_0) \subset Z'_k(\overline{\Omega}, x_0)$.

Indeed, $f_\alpha(x) = \varrho^{k+|\alpha|} D^{|\alpha|} f \in C(\overline{\Omega}) \subset L_1(\Omega)$, if $f \in Z_{-k}(\overline{\Omega}, x_0)$, and then the following linear continuous functionals f_α on $Z_k(\overline{\Omega}, x_0)$ are defined: $(\varphi, f_\alpha) = \int\limits_\Omega \varphi_\alpha f_\alpha dx = \int\limits_\Omega \varrho^{-k+|\alpha|} D^\alpha \varphi f_\alpha dx = \int\limits_\Omega \varrho^{-k+|\alpha|} D^\alpha \varphi \varrho^{k+|\alpha|} D^\alpha f dx = \int\limits_\Omega D^\alpha \varphi \varrho^{2|\alpha|} D^\alpha f dx, \varphi \in Z_k(\overline{\Omega}, x_0)$, for any multi-index α, in particular $(\varphi, f) = \int\limits_{\Omega} \varphi_0 f_0 dx = \int\limits_{\Omega} \varrho^{-k} \varphi \varrho^k f dx = \int\limits_{\Omega} \varphi f dx, \varphi \in Z_k(\overline{\Omega}, x_0)$. Hence if $f \in Z_{-k}(\overline{\Omega}, x_0)$, f will be a regular generalized function of $Z_k'(\overline{\Omega}, x_0)$.

4) Let $g_\alpha \in L_1(\Omega)$ for any multi-index α, then for any $\varphi \in Z_k(\overline{\Omega}, x_0)$ the expressions $\int\limits_{\Omega} \varrho^{-k+|\alpha|} D^\alpha \varphi g_\alpha dx$ exist and for any natural number N we have that the function $g(x) = \sum\limits_{|\alpha| \le N} D^\alpha((-1)^{|\alpha|} g_\alpha \varrho^{-k+|\alpha|})$ (derivatives are regarded in generalized sense) is a linear continuous functional on $Z_k(\overline{\Omega}, x_0)$: $(\varphi, g) = \sum\limits_{|\alpha| \le N} \int\limits_{\Omega} \varrho^{-k+|\alpha|} g_\alpha D^\alpha \varphi dx$. In particular $g(x) = g_0(x)(x - x_0)^{-\kappa} \in Z'_{|\kappa|}(\overline{\Omega}, x_0)$.

5) For any multi-index α, $\varphi \in Z_k(\overline{\Omega}, x_0)$, bounded functions $g_\alpha(x)$ in Ω and any numbers $p_\alpha > -n$, the expression $\int\limits_{\Omega} g_\alpha \varrho^{p_\alpha + |\alpha| - k} D^\alpha \varphi dx$ exists. Then $g(x) = \sum\limits_{|\alpha| \le N} D^\alpha (g_\alpha \varrho^{p_\alpha + |\alpha| - k}) \in Z_k'(\overline{\Omega}, x_0)$.

In particular,

$$g(x) = g_0(x)(x - x_0)^{-\kappa} \in Z'_{|\kappa| - n + \varepsilon}(\overline{\Omega}, x_0)$$

for bounded function $g_0(x)$ in $\overline{\Omega}$ and any $\varepsilon > 0$.

Notice, that $g(x) \in D'(\overline{\Omega})$ for $|\kappa| < n$ and $g(x) \notin D'(\overline{\Omega})$ for $|\kappa| \ge n$.

Let $f_0(x)$ be a bounded function in Ω, $F_0(x) = f_0(x)(x - x_0)^\kappa, |\kappa| \ge 0, F_1 = \cdots = F_m = 0$, $N^* = \{0\}$. We shall obtaine that the solution $u(x)$ of the problem (1), such as $Pu = 0$, belongs to $Z'_{|\kappa| - 2m - n + \varepsilon}(\overline{\Omega},\ x_0)$ for any $\varepsilon > 0$. It follows from the following theorem.

Theorem 1 Let $F_0 \in Z_p'(\overline{\Omega}, x_0), p > 2m - n, F_1 = \cdots = F_m = 0, N^* = \{0\}$, $u(x)$ be such solution of the problem (1) that $Pu(x) = 0$. Then $u(x) \in Z'_{p-2m}(\overline{\Omega}, x_0)$.

This conclusion is exact in the sense that there exists $F_0(x) = Au(x) \in Z_p'(\overline{\Omega}, x_0)$, for the solution $u(x) \in Z'_{p-2m-\varepsilon}(\overline{\Omega}, x_0) \subset Z'_{p-2m}(\overline{\Omega}, x_0), \varepsilon > 0$, of the problem, and it is possibly, that doesn't exist such $F_0 = Au(x) \in Z_p'(\overline{\Omega}, x_0)$, for the solution $u(x) \in Z'_{p-2m+\varepsilon}(\overline{\Omega}, x_0)$. Really, for $u(x) \in Z'_{p-2m+\varepsilon}(\overline{\Omega}, x_0)$, we have $Au(x) \in Z'_{p+\varepsilon}(\overline{\Omega}, x_0)$, and $Z_p'(\overline{\Omega}, x_0) \subset Z'_{p+\varepsilon}(\overline{\Omega}, x_0)$, for $\varepsilon > 0$.

Proof. It is shown in [5] that $\psi(y) = (G_0^* \varphi)(y) = \int\limits_{\Omega} \varphi(x) G(x, y) dx \in X(\overline{\Omega}) \subset D(\overline{\Omega})$ for any $\varphi \in D(\overline{\Omega})$. Let us study its properties for $\varphi \in Z_k(\overline{\Omega}, x_0), x_0 \in \overline{\Omega}$.

Let $h(x) \in D(\Omega), 0 \le h(x) \le 1, h(x) = \begin{cases} 1, & |x - x_0| < \eta \\ 0, & |x - x_0| > 2\eta \end{cases}, \psi(y) = (G_0^* \varphi)(y) = \int\limits_{\Omega} h(x) \varphi(x) G(x, y) dx + \int\limits_{\Omega} (1 - h(x)) \varphi(x) G(x, y) dx = \psi_1(y) + \psi_2(y)$.

The function $(1 - h(y))\psi(y) = 0$, if $|y - x_0| < \eta$, therefore $(1 - h(y))\psi(y) \in Z_{k+2m}(\overline{\Omega}, x_0)$ for any $k \in \mathbb{R}^1$, $\varphi \in Z_k(\overline{\Omega}, x_0)$.

The function $(1 - h(x))\varphi(x) = 0$, if $|x - x_0| < \eta$, therefore also $h(y)\psi_2(y) \in Z_{k+2m}(\overline{\Omega}, x_0)$ for any $\varphi \in Z_k(\overline{\Omega}, x_0)$, $k \in \mathbb{R}^1$.

The function $h(y)\psi_1(y) = O\big(\varrho^{2m+k}(y, x_0)\big)$, if $\varphi \in Z_k(\overline{\Omega}, x_0)$, $k > -n$. We shall obtaine that, for any multi-index α, the function $v_\alpha(y) = \varrho^{|\alpha|}(y, x_0) D^\alpha \int\limits_{\Omega} h(x) \varphi(x) G_0(x, y)$

$h(y)dx = w_\alpha(y)\varrho^{2m+k}(y,x_0)$, where $w_\alpha \in C(\overline{\Omega})$.

Indeed, we assume the function $v_\alpha(y)$ in the form of the sum $v_{1\alpha}(y)+v_{2\alpha}(y)+v_{3\alpha}(y)$ of three items respectively to the partition of the domaine Ω into $\Omega_1 = \{x \in \Omega : |x-x_0| < \frac{|y-x_0|}{2}\}$, $\Omega_2 = \{x \in \Omega : |x-y| < \frac{|y-x_0|}{2}\}$, $\Omega_3 = \Omega \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Later the estimates of the derivatives of $G_0(x,y)$ are using.

As a result, $(G_0^*\varphi)(y) \in Z_{k+2m}(\overline{\Omega},x_0)$, if $\varphi \in Z_k(\overline{\Omega},x_0)$ and $k > -n$, or $(G_0^*\varphi)(y) \in Z_p(\overline{\Omega},x_0)$, for $\varphi \in Z_{p-2m}(\overline{\Omega},x_0)$ and $p > 2m-n$.

Then the map $F_0 \to u$, defined by $(\varphi,u) = (G_0^*\varphi, F_0)$, determines $u \in Z'_{p-2m}(\overline{\Omega},x_0)$ for any $F_0 \in Z'_p(\overline{\Omega},x_0)$, if $p > 2m-n$. Since $G_0^*(A^*\psi) = \psi$, defined by that map function u satisfies the condition (2) for any $\psi \in Z_{p-2m}(\overline{\Omega},x_0)$, and, therefor, any $\psi \in Z_{p-2m}(\overline{\Omega},x_0) \cap X(\overline{\Omega})$.

The spaces $Z_k(S,x_0), x_0 \in S$, are defined similarly and we obtaine that $\psi_j(y) = (G_j^*\varphi)(y) \in Z_{k+m_j+1}(S,x_0), j = \overline{1,m}$, if $\varphi \in Z_k(\overline{\Omega},x_0), k > -n$.

Therefore, the map $F_j \to u_j$, defined by $(\varphi,u_j) =< \int\limits_\Omega \varphi(x)G_j(x,y)dx, F_j >$, determines $u_j \in Z'_{p-m_j-1}(\overline{\Omega},x_0)$ for any $F_j \in Z'_p(S,x_0)$, if $p > m_j+1-n$.

Let $F_0 \in Z'_p(\overline{\Omega},x_0), F_j \in Z'_{p_j}(S,x_j), x_0 \in \overline{\Omega}, x_j \in S, j = \overline{1,m}, \overline{p} = (p,p_1,\ldots,p_m), X_{\overline{p}} = X_{\overline{p}}(\overline{\Omega},x_0,x_1,\ldots,x_m) = \{\varphi(x) \in Z_p(\overline{\Omega},x_0) \cap (\cap_{j=1}^m Z_{p_j}(S,x_j)) : \hat{B}_j\varphi\,|_S = 0\}$ (if $x_i = x_j$, the space $Z_{p_i} \cap Z_{p_j}$ remaines by $Z_k(S,x_i), k = max\{p_i,p_j\}$).

It can be shown that, in the case $N^* = \{0\}$, the function (4) satisfies the equality (2) for any $\psi \in X_{\overline{p}}(\overline{\Omega},x_0,x_1,\ldots,x_m)$, if $p > 2m-n (x_0 \in \Omega), p > max\{2m-n, m''\}(x_0 \in S), p_j > max\{1-n+m_j, m_j+1-2m+m''\}, j = \overline{1,m}$, m'' denotes the maximum of the degrees of $\hat{B}_l, l = \overline{1,m}$.

So, we obtaine that, in the case $N^* = \{0\}$,

$$\begin{gathered} F_0 = f_0(x)(x-x_0)^{-\kappa}, F_j = f_j(x)(x-x_j)^{-\kappa_j}, \\ f_0(x) \in L_\infty(\Omega), f_j(x) \in L_\infty(S) \end{gathered} \tag{5}$$

(then $F_0 \in Z'_{|\kappa|-n+\varepsilon|}(\overline{\Omega},x_0)$, $F_j \in Z'_{|\kappa_j|-n+1+\varepsilon_j}(S,x_j), \varepsilon, \varepsilon_j > 0)$, $j = \overline{1,m}$, $|\kappa| > 2m-\varepsilon(x_0 \in \Omega), |\kappa| > max\{2m, n+m''\} - \varepsilon(x_0 \in S), |\kappa_j| > max\{m_j, n-2m+m_j+m''\} - \varepsilon_j, j = \overline{1,m}$, the formula (4) determines the solution of the problem (1)

$$u(x) = u_0(x) + \sum_{j=1}^m u_j(x),$$

$$u_0(x) \in Z'_{|\kappa|-n+\varepsilon-2m}(\overline{\Omega},x_0), u_j(x) \in Z'_{|\kappa_j|-n-m_j+\varepsilon_j}(\overline{\Omega},x_j),$$

in the sense of fulfilment (2) for any $\psi \in X_{\overline{p}}$, where $p = |\kappa|-n+\varepsilon, p_j = |\kappa_j|-n+1+\varepsilon_j, j = \overline{1,m}$. The existence of such function ψ may be proved.

This solution can have the power singularities of the order $|\kappa|-2m+\varepsilon$ inside Ω, $|\kappa_j|-m_j+1+\varepsilon$ on its boundary $S, j = \overline{1,m}$.

We obtaine the similar results in the case of the power singularities of the right-hand side data on any smooth closed manifold S_1 inside Ω.

There are similar properties of the solutions of some boundary value problems for the elliptic operators in fractional derivatives

$$Au(x) = \frac{1}{(2\pi)^{-n}} \int_{\mathbb{R}^n} a(x,\xi)\mathcal{F}u(\xi)e^{i(x,\xi)}d\xi,$$

where $\mathcal{F}u$ denotes Fourier transform of the function $u(x)$,

$$a(x,\xi) = \sum_{j=1}^{N} \sum_{|\alpha|=s_j \leq s} a_\alpha(x)(-i\xi)^\alpha,$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i, s_j, s are nonnegative numbers, fractional in general, $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$.

We assume the existence of the normal fundamental solution $\omega(x, y) \in C^\infty(x \neq y)$ satisfying the following estimate

$$|\omega(x,y)| \leq \begin{cases} C|x-y|^{s-n}, & n\, is\, odd \\ C|x-y|^{s-n}(ln|x|+1), & n\, is\, even \end{cases}.$$

In the case of constant coefficients such fundamental solution exists.

References

1. .., , . . . **20** (1968), no. 3, 412-417.
2. Roitberg Y., *Elliptic Boundary Value Problems in the Spaces of Distributions. Kluver Academic Publishers.* (1996).
3. .., , ... **36** (1971), no. 1, 202-209.
4. .., , ... **33** (1969), no. 1, 109-137.
5. .., .., , . **22** (1987), no. 3, 518-521.
6. .., .., ., . . . **19** (1967), no. 5, 3-32.

Ivan Franko Lviv State University
Universitetskaya str., 1
290001 Lviv, Ukraine
E-mail: diffeq@ franko.lviv.ua
Tel: 794-593