BIFURCATION OF AN EQUILIBRIUM POINT IN A SYSTEM OF NONLINEAR PARABOLIC EQUATIONS WITH TRANSFORMED ARGUMENT

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1.Introduction. We consider the system of nonlinear parabolic equations with transformed argument

$$\frac{\partial u}{\partial t} = D(t,\varepsilon)\frac{\partial^2 u}{\partial x^2} + A(t,\varepsilon)u + B(t,\varepsilon)u_\Delta + f(t,u,u_\Delta,\varepsilon) \tag{1}$$

with periodic condition

$$u(t,x+2\pi) = u(t,x). \tag{2}$$

Here $u_\Delta = u(t,x-\Delta), \Delta$ is a transformation of the argument, the matrices $D(t,\varepsilon)$, $A(t,\varepsilon)$, $B(t,\varepsilon)$ and function $f : R^{2n+p+1} \to R^n$ are fourfold continuously differentiable with respect to all arguments and 2π - periodic with respect to t, $f(t,u,v,\varepsilon) = O(|u|^2 + |v|^2)$ as $|u| + |v| \to 0$. Therefore the function $f(t,u,v,\varepsilon)$ satisfies to the conditions

$$f(t,0,0,\varepsilon) = 0, |f(t,u,v,\varepsilon) - f(t,u',v',\varepsilon)| \le \nu(|u-u'|^2 + |v-v'|^2)^{1/2},$$

$$|u| \le \rho, |u'| \le \rho, |v| \le \rho, |v'| \le \rho, \tag{3}$$

where $|u|^2 = u_1^2 + \ldots + u_n^2$, Lipschitz constant ν may be make sufficiently small under decreasing ρ. Function $f(t,u,v,\varepsilon)$ can be determined outside the region $|u| \le \rho, |v| \le \rho$, so that the conditions (3) valid overall the space. Let the matrix $D(t,\varepsilon)$ is positive definite.

System (1) is used for modelling of nonlinear effects in optics [1]. The autonomous parabolic equation with transformed argument was considered in paper [2].

We consider the linear system

$$\frac{\partial u}{\partial t} = D(t,\varepsilon)\frac{\partial^2 u}{\partial x^2} + A(t,\varepsilon)u + B(t,\varepsilon)u_\Delta. \tag{4}$$

We will search the solution of the problem (4),(2) in the form of complex Fourier series

$$u(t,x) = \sum_{k=-\infty}^{\infty} y_k(t) exp(-ikx), y_{-k}(t) = \bar{y}_k(t). \tag{5}$$

Substituting (5) into (4) and comparising the coefficients under $exp(-ikx)$, we obtain the countable system of differential equations in Fourier coefficients

$$\frac{dy_k(t)}{dt} = [-k^2 D(t,\varepsilon) + A(t,\varepsilon) + B(t,\varepsilon) exp(ik\Delta)] y_k(t), k = 0, \pm 1, \dots \quad (6)$$

System (6) is a one of linear differential equations with periodic coefficients. According to the Floquet theorem, a matrix $H_k(t,\varepsilon)$, $det H_k(t,\varepsilon) \neq 0$, $H_k(t+2\pi,\varepsilon) = H_k(t,\varepsilon)$ exists, such that the substitution $y_k = H_k(t,\varepsilon) z_k$ transforms system (6) to the form

$$\frac{dz_k}{dt} = C_k(\varepsilon) z_k, C_{-k}(\varepsilon) = \bar{C}_k(\varepsilon), k = 0, \pm 1, \dots \quad (7)$$

Suppose that the characteristic equation $det(C_k(\varepsilon) - \lambda E) = 0, k \in Z$, has the simple roots $\alpha_m(\varepsilon) \pm i\beta_m(\varepsilon), \alpha_m(0) = 0, \beta_m(0) = \lambda_m > 0, m = 1, ..., p$, and the remaining of roots satisfies to the condition $|\,Re\,\lambda| > \gamma + \delta, \gamma > \delta > 0$. Suppose that ε is the p - dimensional parameter.

We will search the solution of the problem (1),(2) in the form of series (5). Substituting (5) into (1) and comparising the coefficients under $\exp(-ikx)$, $k \in Z$, we obtain the countable system of differential equations in Fourier coefficients

$$\frac{dy}{dt} = M(t,\varepsilon) y + F(t,y,\varepsilon), \quad (8)$$

where $y = (y_0, y_1, y_{-1}, ...)^T$, $M(t,\varepsilon)$ is infinite blocks- diagonal matrix with the blocks $M_k(t,\varepsilon) = -k^2 D(t,\varepsilon) + A(t,\varepsilon) + B(t,\varepsilon)\exp(ik\Delta)$, $k = 0, \pm 1, ...;$ $F(t,y,\varepsilon) = (f_0, f_1, f_{-1}, ...)^T$ is nonlinear function, where f_k are the Fourier coefficients of the function $f(t,u,u_\Delta,\varepsilon)$ under $\exp(-ikx)$.

We will show that the function $F(t,y,\varepsilon)$ satisfies to the Lipschitz condition. Let us introduse in the space of sequences the following norm $|y| = (\sum_{k=-\infty}^{\infty} |y_k|^2)^{1/2}$. We consider another vector $z = (z_0, z_1, z_{-1}, ...)^T$ of Fourier coefficiets for solution $v(t,x)$ of equation (1) and the corresponding vector $F(t,z,\varepsilon) = (g_0, g_1, g_{-1}, ...)^T$. Using Parseval equation, we obtain

$$|F(t,y,\varepsilon) - F(t,z,\varepsilon)| = (\sum_{k=-\infty}^{\infty} |f_k - g_k|^2)^{1/2} = (\frac{1}{2\pi}\int_0^{2\pi} |f(t,u,u_\Delta,\varepsilon) -$$

$$-f(t,v,v_\Delta,\varepsilon)|^2 dx)^{1/2} \leq \nu(\frac{1}{2\pi}\int_0^{2\pi} (|u-v|^2 + |u_\Delta - v_\Delta|^2) dx)^{1/2} =$$

$$= \nu(2\sum_{k=-\infty}^{\infty} |y_k - z_k|^2)^{1/2} = \sqrt{2}\nu|y-z|.$$

Therefore the function F satisfies the Lipschitz condition with the constant $\sqrt{2}\nu$.

In the system (8), we make the substitution $y_k = H_k(t,\varepsilon) z_k$, $k = 0, \pm 1, ...$, then we obtain the system

$$\frac{dz}{dt} = C(\varepsilon) z + G(t,z,\varepsilon), \quad (9)$$

where $z = (z_0, z_1, z_{-1}, ...)^T$, $C(\varepsilon) = diag(C_0(\varepsilon), C_1(\varepsilon), C_{-1}(\varepsilon), ...)$, $G(t,z,\varepsilon) = H^{-1}(t,\varepsilon) F(t, H(t,\varepsilon) z, \varepsilon)$, $H(t,\varepsilon) = diag\,(H_0, H_1, H_{-1}, ...)$. We reduce the matrices $C_k(\varepsilon)$ with eigenvalues $\alpha_m(\varepsilon) \pm i\beta_m(\varepsilon)$ and eigenvalues with positive real parts to the Jordan canonical form. Under this transformation, we obtain the system

$$\frac{dw_1}{dt} = A_1(\varepsilon)w_1 + G_1(t, w, \varepsilon),$$

$$\frac{dw_2}{dt} = A_2(\varepsilon)w_2 + G_2(t, w, \varepsilon), \tag{10}$$

where $w = (w_1, w_2)^T$, $A_1(\varepsilon) = diag(A_3(\varepsilon), A_4(\varepsilon))$, $w_1 \in R^{l+2p}$, w_2 belong to the some Banach space M, the eigenvalues of the matrix $A_3(\varepsilon)$ lie on the half-plane $Re\,\lambda > \gamma + \delta$, $A_4(\varepsilon)$ is the diagonal matrix with number $\alpha_m(\varepsilon) \pm i\beta_m(\varepsilon)$ on diagonal, and the eigenvalues of the infinite blocks-diagonal matrix $A_2(\varepsilon)$ lieing on half-plane $Re\,\lambda < -\gamma - \delta$. Since the vector-function G satisfies the Lipschitz condition and $G(t, 0, \varepsilon) = 0$, we obtain

$$G_1(t, 0, \varepsilon) = G_2(t, 0, \varepsilon) = 0, (|G_1(t, v, \varepsilon) - G_1(t, w, \varepsilon)|^2+$$

$$+|G_2(t, v, \varepsilon) - G_2(t, w, \varepsilon)|^2)^{1/2} \leq \nu_1|v - w|. \tag{11}$$

The following estimations are valid

$$|\exp[A_3(\varepsilon)t] \leq N\exp[(\gamma + \delta)t], t \leq 0, |\exp[A_2(\varepsilon)t]| \leq$$

$$\leq N\exp[-(\gamma + \delta)t], t \geq 0, |\exp[A_4(\varepsilon)t]| \leq N\exp[(\gamma - \delta)|t|], t \in R. \tag{12}$$

2.Existence and properties of integral manifolds.

Theorem 1. Let the estimates (11),(12) holds. Thus, if

$$\nu_1 < \frac{\delta}{N(1 + 2N)}, \tag{13}$$

then there exists a function $h : R^{l+3p+1} \to M$,

$$h(t, 0, \varepsilon), |h(t, w_1, \varepsilon) - h(t, w_1', \varepsilon)| \leq \frac{1}{2}|w_1 - w_1'|, \tag{14}$$

such that the set $S^- = \{(t, w_1, w_2)|t \in R, w_1 \in R^{l+2p}, w_2 = h(t, w_1, \varepsilon), w_2 \in M\}$ is the integral manifold of the system (10). For any solution $w(t) = (w_1(t), h(t, w_1(t), \varepsilon))$ of the system (10) on manifold S^-, the following estimate is valid

$$|w(t)| \leq 2N|w_1(\sigma)|\exp[\gamma(\sigma - t)], t \leq \sigma.$$

Theorem 2. Let the conditions (11)-(13) are satisfied. Then there exists a function $g : R^{p+1} \times M \to R^{l+2p}$, $g(t, 0, \varepsilon) = 0$, $|g(t, w, \varepsilon) - g(t, w', \varepsilon)| \leq \frac{1}{2}|w - w'|$, such that the set $S^+ = \{(t, w_1, w_2)|t \in R, w_2 \in M, w_1 = g(t, w_2, \varepsilon), w_1 \in R^{l+2p}\}$ is the integral manifold of the system (10). For any solution $w(t) = (g(t, w_2(t), \varepsilon), w_2(t))$ of the system

(10) on manifold S^+, the following estimate is valid $|w(t)| \leq 2N|w_2(\sigma)|exp[\gamma(\sigma - t)]$, $t \geq \sigma$.

Let $t = \sigma$ is some number (initial value). We show that the integral manifold S^- is exponential stable.

Note that the equation on manifold S^- is of the following form

$$\frac{dv}{dt} = A_1(\varepsilon)v + G_1(t, v, h(t, v, \varepsilon), \varepsilon). \tag{15}$$

Theorem 3. Let $w(t) = (w_1(t), w_2(t))$ be arbitrary solution of the system (10) with initial value $w(\sigma)$ under $t = \sigma$. If the condition (13) is satisfied, then there exists a solution $\xi(t) = (v(t), h(t, v(t), \varepsilon)$ on manifold S^-, such that the following estimate is valid

$$|w(t) - \xi(t)| \leq 2N|w_2(\sigma) - h(\sigma, v(\sigma), \varepsilon)|exp[\gamma(\sigma - t)], t \geq \sigma.$$

The equation (15) can be represented in the form

$$\frac{dw_3}{dt} = A_3(\varepsilon)w_3 + G_3(t, w_3, w_4, h(t, w_3, w_4, \varepsilon), \varepsilon),$$

$$(16)$$

$$\frac{dw_4}{dt} = A_4(\varepsilon)w_4 + G_4(t, w_3, w_4, h(t, w_3, w_4, \varepsilon), \varepsilon).$$

where $v = (w_3, w_4)$, $G_1 = (G_3, G_4)$. If the condition (13) is satisfies, then the integral manifold $S_1^+ = \{(t, w_3, w_4)|t \in R, w_4 \in R^{2p}, w_3 = r(t, w_4, \varepsilon), w_3 \in R^l\}$ of the system (16) exists [3,4]. The function $r(t, w, \varepsilon)$ satisfies the following estimate

$$r(t, 0, \varepsilon) = 0, |r(t, w, \varepsilon) - r(t, v, \varepsilon)| \leq \frac{1}{2}|w - v|, w \in R^{2p}, v \in R^{2p}.$$

We denote $r_1(t, w, \varepsilon) = h(t, r(t, w, \varepsilon), w, \varepsilon)$.

Theorem 4. Let the conditions (11)-(13) be satisfied. Then there exists the central manifold $S = \{(t, w_3, w_4, w_2)|t \in R, w_4 \in R^{2p}, w_3 = r(t, w_4, \varepsilon), w_3 \in R^l, w_2 = r_1(t, w_4, \varepsilon), w_2 \in M\}$ of the system (10).

3.Bifurcation of equilibrium point. The equation on manifold S is of the following form

$$\frac{dw_4}{dt} = A_4(\varepsilon)w_4 + G_4(t, r(t, w_4, \varepsilon), w_4, r_1(t, w_4, \varepsilon), \varepsilon). \tag{17}$$

The equation (17) can be represented in the form

$$\frac{dv_k}{dt} = [\alpha_k(\varepsilon) + i\beta_k(\varepsilon)]v_k + V_k(t, v, \bar{v}, \varepsilon),$$

$$(18)$$

$$\frac{d\bar{v}_k}{dt} = [\alpha_k(\varepsilon) - i\beta_k(\varepsilon)]\bar{v}_k + \bar{V}_k(t, v, \bar{v}, \varepsilon),$$

where v_k is the complex variable, $v = (v_1, ..., v_p)^T$, $V_k(t + 2\pi, v, \bar{v}, \varepsilon) = V_k(t, v, \bar{v}, \varepsilon)$, $V_k(t, v, \bar{v}, \varepsilon) = O(|v|^2)$ as $|v| \to 0$, $k = 1, ..., p$.

Let the following condition be satisfied

1) $n_1\lambda_1 + ... + n_p\lambda_p \neq m$ as $0 < |n_1| + ... + |n_p| < 6$, where $m, n_1, ..., n_p$ are the integer numbers.

Substituting the variable

$$v = x + \sum_{k=2}^{4} W_k(t, x, \bar{x}, \varepsilon),$$

where W_2, W_3, W_4 are the forms of the 2,3 and 4 order with periodic coefficients, we transform the system (18) to the following form [5,6]

$$\frac{dx_k}{dt} = [\alpha_k(\varepsilon) + i\beta_k(\varepsilon)]x_k + x_k \sum_{j=1}^{p} a_{kj}(\varepsilon)x_j\bar{x}_j + X_k(t, x, \bar{x}, \varepsilon),$$

$$\frac{d\bar{x}_k}{dt} = [\alpha_k(\varepsilon) - i\beta_k(\varepsilon)]\bar{x}_k + \bar{x}_k \sum_{j=1}^{p} \bar{a}_{kj}(\varepsilon)x_j\bar{x}_j + \bar{X}_k(t, x, \bar{x}, \varepsilon),$$

where $X_k(t + 2\pi, x, \bar{x}, \varepsilon) = X_k(t, x, \bar{x}, \varepsilon)$, $X_k(t, x, \bar{x}, \varepsilon) = O(|x|^5)$ as $|x| \to 0$. Passing to the polar coordinates $x_k = r_k \exp(i\varphi_k)$, $\bar{x}_k = r_k \exp(-i\varphi_k)$, we obtain the real system

$$\frac{dr_k}{dt} = \alpha_k(\varepsilon)r_k + r_k \sum_{j=1}^{p} b_{kj}(\varepsilon)r_j^2 + R_k(t, r, \varphi, \varepsilon),$$

$$\frac{d\varphi_k}{dt} = \beta_k(\varepsilon) + \sum_{j=1}^{p} c_{kj}(\varepsilon)r_j^2 + \Phi_k(t, r, \varphi, \varepsilon),$$

where $b_{kj}(\varepsilon) = Re\, a_{kj}(\varepsilon)$, $c_{kj}(\varepsilon) = Im\, a_{kj}(\varepsilon)$, $R_k(t, r, \varphi, \varepsilon) = O(|r|^5)$, $\Phi_k(t, r, \varphi, \varepsilon) = O(|r|^4)$ as $|r| \to 0$.

We consider the bifurcation equation $B(\varepsilon)r^2 + a(\varepsilon) = 0$, where $B(\varepsilon)$ is the matrix with elements $b_{kj}(\varepsilon)$, $a(\varepsilon)$ and r^2 are the vectors with elements $\alpha_k(\varepsilon)$ and r_j^2.

Theorem 5. Let $det B(0) \neq 0, det \frac{da}{d\varepsilon}(0) \neq 0$, the all elements of vector $B^{-1}(\varepsilon)a(\varepsilon)$ are negative and condition 1 is satisfied. Then, there exists an invariant torus of the system (1).

The solutions on the torus are quasi-periodic if $|(n, \lambda) + q| > \gamma|n|^{-p-1}, \lambda = (\lambda_1, ..., \lambda_p) = (\beta_1(0), ..., \beta_p(0))$, where γ is some positive number, $n = (n_1, ..., n_p)$, $q, n_1, ..., n_p$ are integer numbers.

References

1. Akhmanov S.A., Vorontsov M.A., *Instabilities and structures in coherent nonlinear optical systems*, Nonlinear waves. Dynamics and evolution. Moskow:Nauka (1989), 228-237.
2. Kashchenko S.A., *Asymptotic space-ihomogeneous structures in coherent nonlinear-optical systems*, J. Vychisl. Math. i Math. Phys. **31** (1991), no. 3, 467-473.
3. Plis V.A., *Integral sets of periodic systems of differential equations*, Moskow: Nauka (1977), 304.

4. Fodchuk V.I., Klevchuk I.I., *Integral sets and reduction principle for differential-functional equations*, Ukr. Math. J. **34** (1982), no. 3, 334-340.
5. Samoilenko A.M., Polesya I.V., *The birth of the invariant sets in a neigborhood of equilibrium point*, Differentsial'nye Uravneniya **11** (1975), no. 8, 1409-1415.
6. Bibikov Yu.N., *Hopf bifurcation for quasi-periodic motions*, Differentsial'nye Uravneniya **16** (1980), no. 9, 1539-1544.

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