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ScienceDirect
Nonlinear Analysis 69 (2008) 53–72
Nonlinear Analysis
www.elsevier.com/locate/na

On the stability of invariant sets of systems with impulse effect

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Received 17 August 2006; accepted 26 April 2007

Abstract

A system of differential equations with impulse effect is considered. It is assumed that this system has an invariant set M. By means of the direct Lyapunov method, the necessary and sufficient conditions of its uniform asymptotic stability are obtained. The conditions on the perturbations of right hand sides of differential equations and impulse effects, under which the uniform asymptotic stability of the invariant set M of the "nonperturbed" system implies the uniform asymptotic stability of the invariant set of the "perturbed" system, are obtained. The stability properties of invariant sets of periodic systems are also studied.

Keywords: Differential equations with impulse effect; Stability; Lyapunov's direct method

1. Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [2,1,14,15,37,45,48,53,54]. In recent years, the study of impulsive systems has received an increasing interest [10,21,30,45].

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. Consequently, the theory of impulsive differential equations is interesting in itself and has many applications. Thus there is every reason for studying the theory of impulsive differential equations as a well deserved discipline. Impulsive differential equations consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset. Since impulsive systems can involve impulses at variable times, they are in general time-varying systems wherein the resetting events are both a function of time

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.04.040

and the system's state. In the case where the resetting events are defined by a prescribed sequence of times which are independent of the system state, the equations are known as time-dependent differential equations [3,4,11,17,19,30]. Alternatively, in the case where the resetting events are defined by a manifold in the state space that is independent of time, the equations are autonomous and are known as state-dependent differential equations [3,4,11,17,19,30].

Consider the system of time-dependent differential equations with impulse effect at fixed moments of time:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x), \quad t \neq \tau_i, t \in \mathbb{R}_+ \tag{1.1}$$

$$x(\tau_i^+) - x(\tau_i) = J_i(x), \quad i \in \mathbb{N} \tag{1.2}$$

where $t \in \mathbb{R}_+ := [0, \infty)$ is time, $x \in \mathbb{R}^n$, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots,\ \tau_i \to \infty$, $\mathbb{N}$ is the set of positive integers, $x(\tau_i^+)$ means the right-hand limit of x at τ_i.

Let $\Phi(t, t_0, x_0)$ be any solution of system (1.1) and (1.2) starting at (t_0, x_0) where $t_0 \in (\tau_0, \tau_1)$. The explanation of the behavior of solutions of an system of impulsive differential equations one can find in [3,22,30,45]. It is assumed that a solution $\Phi(t, t_0, x_0)$ of a system of differential equations with impulse effect is a left continuous function at the instants of impulse effect, i.e. $\Phi(\tau_k^-, t_0, x_0) = \Phi(\tau_k - 0, t_0, x_0) = \Phi(\tau_k, t_0, x_0)$.

One of the main problems in the investigation of dynamical systems (in particular, systems with impulse effect) is the stability problem. If one needs to study the stability of some solution $\Phi(t, t_0, x_0)$ of system (1.1) and (1.2), this problem can be reduced to the problem of the stability investigation of the zero solution of the system of perturbed motion. So we can assume that $X(t, 0) \equiv 0$, $J_i(0) = 0$, and system (1.1) and (1.2) has the trivial solution

$$x = 0. \tag{1.3}$$

The problem of the stability of solution (1.3) of system (1.1) and (1.2) was investigated by many authors [3,6,16,18, 22,26–32,45,49].

Gurgula and Perestyuk [16] proposed to use Lyapunov's direct method for stability investigation of solution (1.3) of system (1.1) and (1.2). They obtained sufficient conditions of asymptotic stability of the zero solution of system (1.1) and (1.2) by means of a Lyapunov function $V(t, x)$. Bainov and Simeonov [3] proved that for some classes of systems of form (1.1) and (1.2) there exists a Lyapunov function V which satisfies conditions of the Gurgula–Perestyuk theorem.

One of the most basic issues in system theory is stability of dynamical systems. System stability is characterized by analysing the response of a dynamical system to small perturbations in the system states. Specifically, an equilibrium point of a dynamical system is said to be stable if, for small values of initial distubances, the perturbed motion remains in an arbitrarily prescribed small region of the state space. Nowadays one can observe a growing interest among researchers in problems of the property of perturbed motions to remain near multidimentional geometric objects (goal sets, submanifolds) and the relevant phenomena of invariance and attractivity of these multidimentional geometric objects. This is reflected in the development of the new sections of nonlinear control theory, nonlinear mechanics, theory of oscillating processes and robotics. Lyapunov's direct method can be applied to the investigation of the stability of invariant and integral sets of impulsive differential systems. Note that in certain cases, the investigation of the stability of invariant or integral sets of impulsive systems of general form (where τ_i depend on x in (1.1) and (1.2)) can be reduced to the investigation of the stability of an invariant or integral set of a certain system with fixed sequence of times of impulse influence [34,41].

The set $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ is an integral set of system (1.1) and (1.2) if, for any point $(t_0, x_0) \in M$, it follows that $(t, \Phi(t, t_0, x_0)) \in M$ for $t \geq t_0$ where $\Phi(t, t_0, x_0)$ is a solution of (1.1) and (1.2). Let $M(t) = \{x \in \mathbb{R}^n : (t, x) \in M\}$. In [27,34,39,41] system (1.1) and (1.2) has been considered. It is assumed that there is a continuously differentiable function $V(t, x)$ defined in the domain $\{(t, x) : t \in \mathbb{R}_+, x \in D \subset \mathbb{R}^n\}$ and possessing the properties

$$V(t, x) = 0, \quad (t, x) \in M, \tag{1.4}$$

$$V(t, x) \geq a(\rho(x, M(t))) \tag{1.5}$$

where $a(s)$ is a continuous increasing function, $a(0) = 0$. The typical results obtained in [17,27,29,34,39,41] are the following.

- If
$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \langle \mathbf{grad}_x V, X\rangle \le 0,$$
$$V(\tau_i, x + J_i(x)) \le V(\tau_i, x), \quad i = 1, 2, \ldots,$$
then M is a stable integral set of system (1.1) and (1.2).
- Suppose that, for system (1.1) and (1.2) there exists a continuously differentiable function $V(t, x)$ satisfying conditions (1.4) and (1.5) and such that
$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \langle \mathbf{grad}_x V, X\rangle \le -\phi(V(t, x)),$$
$$V(\tau_i, x + J_i(x)) \le \psi(V(\tau_i, x)), \quad i = 1, 2, \ldots$$
where ϕ and ψ are continuous functions $\mathbb{R}_+ \to \mathbb{R}_+$ with certain properties. Then M is an asymptotically stable integral set of system (1.1) and (1.2).

If $M(t) = M_0 \subset \mathbb{R}^n$ for all $t \ge t_0$, then M_0 is an invariant set of system (1.1) and (1.2) and the above conditions are the sufficient conditions of stability and asymptotic stability of M_0 [11,17,19,27,35,34,39,51]. Note that the stability of integral and invariant sets of ODE was investigated in [5,24,25,36,44].

In [11,19,34,40] the system of state-dependent differential equations with impulse effect

$$\frac{dx}{dt} = f(x) \quad \text{for } x \notin \Gamma, \qquad \Delta x|_{x\in\Gamma} = I(x) \tag{1.6}$$

is considered. Here $x \in \overline{D} \subset \mathbb{R}^n$; D is a bounded domain in $\mathbb{R}^n$, Γ is a surface without contact for the system $dx/dt = f(x)$, $\Gamma \subset \overline{D}$. It has been proved that if there exists a function $V(x)$ which satisfies conditions

$$V(x) = 0 \quad \text{for } x \in M_0,\ M_0 \subset D,$$
$$V(x) > 0 \quad \text{for } x \in \overline{D} \setminus M_0,$$

and, futhermore,

$$\langle \mathbf{grad}_x V, f\rangle \le 0, \quad x \notin \Gamma,$$
$$V(x + I(x)) \le V(x), \quad x \in \Gamma,$$

then the set M_0 is the stable invariant set of system (1.6). If, in addition

$$\langle \mathbf{grad}_x V, X\rangle \le -\varphi(V(x)), \quad x \notin \Gamma,$$

where $\varphi(s)$ is a continuous function, $s \ge 0$, $\varphi(0) = 0$, and $\varphi(s) > 0$ for $s > 0$, then M_0 is an asymptotically stable invariant set of system (1.6).

Assume that $x = (y, z)$ in (1.1) and (1.2) where $y \in \mathbb{R}^l$, $z \in \mathbb{R}^m$, $(l + m = n)$. Unlike in the above papers where the stability of the bounded invariant set $M_0 \subset \mathbb{R}^n$ of system (1.1) and (1.2) is studied, papers [35,47] deal with partial stability (y-stability) of the zero solution of (1.1) and (1.2). Roughly speaking, this means that $\|x_0\|$ is small implies that $\|y(t)\|$ is small. The goal of this paper is to investigate the stability of the set $y = 0$ which is an unbounded set of $\mathbb{R}^n$ (roughly speaking, this means that $\|y_0\|$ is small implies that $\|y(t)\|$ is small). The paper is organized as follows. In Section 2, we introduce the main definitions and prove the theorem on uniform asymptotic stability. Then in Section 3, we state and prove a fundamental result (Theorem 3.1) on the conversion of the theorem about uniform asymptotic stability. Section 4 deals with stability of invariant sets of perturbed systems. Finally, in Section 5, the stability of invariant sets of periodic systems is considered.

2. Preliminaries and main definitions. Theorem on uniform asymptotic stability

The notions of the invariance of multidimensional sets and attractivity of smooth submanifolds (hypersurfaces) are closely connected with the concepts of partial stability. The latter implies that only a part of the system variables or a certain function of the state coordinates tends to a desired (partial, or relative) equilibrium. The concepts of partial stability, going back to famous works by A.M. Lyapunov, E.J. Routh and V. Volterra, were developed in [12,13,23,35,36,38,42,43,46,50].

The concepts of partial stability and set stability, being very similar, are not, in general, identical. Consider system with impulse effect (1.1) and (1.2) where $x = (y, z)$, $y = (y^1, y^2, \dots, y^l) \in \mathbb{R}^l$, $z = (z^1, z^2, \dots, z^m) \in \mathbb{R}^m$ $(l+m = n)$. Denote $X = (Y, Z)$, $Y = (Y_1, Y_2, \dots, Y_l) \in \mathbb{R}^l$, $Z = (Z_1, Z_2, \dots, Z_m) \in \mathbb{R}^m$, $J_i = (J_{yi}, J_{zi})$, $J_{yi} = (J^1_{yi}, J^2_{yi}, \dots, J^l_{yi}) \in \mathbb{R}^l$, $J_{zi} = (J^1_{zi}, J^2_{zi}, \dots, J^m_{zi}) \in \mathbb{R}^m$, and rewrite system (1.1) and (1.2) in the form

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}t} = Y(t, y, z), \quad t \neq \tau_i, t \in \mathbb{R}, \\ &y(\tau_i^+) - y(\tau_i) = J_{yi}(y, z), \quad i \in \mathbb{N}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\frac{\mathrm{d}z}{\mathrm{d}t} = Z(t, y, z), \quad t \neq \tau_i, t \in \mathbb{R}, \\ &z(\tau_i^+) - z(\tau_i) = J_{zi}(y, z), \quad i \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Denote $x_0 = (y_0, z_0)$, $\Phi(t, t_0, x_0) = (F(t, t_0, x_0), E(t, t_0, x_0))$ $(F \in \mathbb{R}^l, E \in \mathbb{R}^m)$ is a solution of system (2.1) and (2.2) satisfying the identities $F(t_0, t_0, x_0) = y_0$, $E(t_0, t_0, x_0) = z_0$ where $x_0 \in \mathbb{R}^n$ if $t_0 \in \bigcup_{i=1}^{\infty}(\tau_{i-1}, \tau_i)$. If, however, $t_0 = \tau_i$, $i \in \mathbb{N}$, then we denote by $\Phi(t, t_0, x_0) = (F(t, t_0, x_0), E(t, t_0, x_0))$ for $t > t_0$ the solution of system (2.1) and (2.2) such that

$$\begin{aligned} &F(t_0^+, t_0^+, x_0 + J_i(x_0)) = y_0 + J_{yi}(x_0), \\ &E(t_0^+, t_0^+, x_0 + J_i(x_0)) = z_0 + J_{zi}(x_0). \end{aligned}$$

According to the existing tradition [3,30,45], the solution $x(t)$ is assumed to be continuous from the left at the points τ_i : $\Phi(\tau_i, t_0, x_0) = \Phi(\tau_i^-, t_0, x_0)$ $i \in \mathbb{N}$.

Suppose that $Y(t, 0, z) \equiv 0$, $J_{yi}(0, z) \equiv 0$. Under these assumptions, system (2.1) and (2.2) has the invariant set M which is described by the equality

$$y = 0. \tag{2.3}$$

This means that $y_0 = 0$ implies $F(t, t_0, x_0) \equiv 0$.

Denote $\|y\| = \max_{1 \le s \le l} |y^s|$, $\|z\| = \max_{1 \le j \le m} |z^j|$,

$$B_H = \{x \in \mathbb{R}^n : \|y\| \le H, \|z\| < \infty\}, \quad G = \bigcup_{i=1}^{\infty}(\tau_{i-1}, \tau_i) \times B_H.$$

Let us make the following assumptions.

(P1) A function $X = (Y, Z)$ is continuous in each domain $(\tau_{i-1}, \tau_i) \times B_H$ $(i \in \mathbb{N})$, $Y(t, 0, z) \equiv 0$,

$$\|Y(t, y_1, z_1) - Y(t, y_2, z_2)\| \le L\|y_1 - y_2\|,$$

and for any $k \in \mathbb{N}$ there exist the finite limits

$$\lim_{t \to \tau_k^-} X(t, x) = X(\tau_k, x), \qquad \lim_{t \to \tau_k^+} X(t, x) = X(\tau_k^+, x).$$

(P2) Functions $J_i = (J_{yi}, J_{zi})(i \in \mathbb{N})$ are continuous, $J_{yi}(0, z) = 0$, $i \in \mathbb{N}$, $z \in \mathbb{R}^m$, and

$$\|J_{yi}(y_1, z_1) - J_{yi}(y_2, z_2)\| \le L\|y_1 - y_2\|, \quad (y_1, z_1) \in B_H, (y_2, z_2) \in B_H, i \in \mathbb{N}.$$

(P3) Constants $\tau_i (i \in \mathbb{N})$ satisfy conditions $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, $\lim_{i \to \infty} \tau_i = +\infty$, and in each segment $[t, \tau] \subset \mathbb{R}_+$, there are no more than p points τ_i where p depends on the length of the segment $[t, \tau]$.

Let us introduce some necessary definitions.

Definition 2.1. The invariant set (2.3) of system (2.1) and (2.2) is said to be stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|y_0\| < \delta$ implies $\|F(t, t_0, x_0)\| < \varepsilon$ for $t \ge t_0$. If δ can be chosen independent of t_0 (i.e. $\delta = \delta(\varepsilon)$), then the invariant set (2.3) is called uniformly stable.

Definition 2.2. The invariant set M of system (2.1) and (2.2) is said to be attractive if for any $t_0 \in \mathbb{R}_+$ there exists an $\eta = \eta(t_0) > 0$, and for any $\varepsilon > 0$ and $x_0 \in B_\eta$ there exists a $\sigma = \sigma(\varepsilon, t_0, x_0) > 0$ such that $\|F(t, t_0, x_0)\| < \varepsilon$ for all $t \ge t_0 + \sigma$. We say that B_η is contained in the domain of attraction of M at the moment t_0.

In other words, M is an attractive set if

$$\lim_{t\to\infty} \|F(t, t_0, x_0)\| = 0. \tag{2.4}$$

Definition 2.3. The set M is said to be uniformly attractive one for system (2.1) and (2.2) if for some $\eta > 0$ and any $\varepsilon > 0$ there exists a $\sigma = \sigma(\varepsilon) > 0$ such that $\|F(t, t_0, x_0)\| < \varepsilon$ for all $t_0 \in \mathbb{R}_+,\ x_0 \in B_\eta$ and $t \geq t_0 + \sigma$.

In other words, M is a uniformly attractive set if the limit relation (2.4) holds uniformly with respect to $t_0 \geq 0,\ x_0 \in B_\eta$.

Definition 2.4. The invariant set M of system (2.1) and (2.2) is said to be:

- asymptotically stable if it is stable and attractive;
- uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

Let $\mathcal{K}$ denote the class of Hahn functions [42], that is $g \in \mathcal{K}$ if $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function such that $g(0) = 0$. Note that in [9,7,8,20] these functions are called *wedges*.

Definition 2.5. We say that a function $V : \mathbb{R}_+ \times B_H \to \mathbb{R}$ belongs to the class $\mathcal{V}_0$ if it is continuous on the set G, satisfies the condition $|V(t, x_1) - V(t, x_2)| \leq L\|y_1 - y_2\|, x_1 = (y_1, z_1), x_2 = (y_2, z_2)$ uniformly with respect to $t \in \mathbb{R}_+, z_1 \in \mathbb{R}^m, z_2 \in \mathbb{R}^m$; $V(t, 0, z) \equiv 0$ for $t \in \mathbb{R}_+, z \in \mathbb{R}^m$, and for any $k \in \mathbb{N}$ there exist the finite limits

$$\lim_{t\to\tau_k^-} V(t, x) = V(\tau_k, x), \qquad \lim_{t\to\tau_k^+} V(t, x) = V(\tau_k^+, x).$$

We say that a function $V \in \mathcal{V}_0$ belongs to the class $\mathcal{V}_1$ if it is a C^1 function on G. For $(t, x) \in G$ we define the derivative of the function $V \in \mathcal{V}_1$ as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot X(t, x).$$

Theorem 2.1. *Let system (2.1) and (2.2) be such that there exists a function $V \in \mathcal{V}_0$ such that*

$$V(t, x) \geq g(\|y\|), \quad t \in \mathbb{R}_+, x \in B_H, g \in \mathcal{K}, \tag{2.5}$$

$$V(t, x) \leq b(\|y\|), \quad t \in \mathbb{R}_+, x \in B_H, b \in \mathcal{K}, \tag{2.6}$$

$$D^+V(t, x) \leq -c(\|y\|), \quad t \in \mathbb{R}_+, x \in B_H, t \neq \tau_i\ (i \in \mathbb{N}), c \in \mathcal{K} \tag{2.7}$$

where $D^+V(t, x)$ is the right upper Dini derivative of the function V along the solution $x(t)$, and

$$V(\tau_i^+, x + J_i(x)) \leq V(\tau_i, x), \quad i \in \mathbb{N}, x \in B_H. \tag{2.8}$$

Then M is an invariant and uniformly asymptotically stable set of system (2.1) and (2.2) and there exists an $H_0 > 0$ $(H_0 < H)$ such that the domain of attraction of M contains the set B_{H_0}. The identities

$$Y(t, 0, z) \equiv 0, \qquad J_{yi}(0, z) \equiv 0, \quad i \in \mathbb{N} \tag{2.9}$$

are also valid.

Proof. If $x_0 \in M$, then $V(t, \Phi(t, t_0, x_0)) \equiv 0$ in view of (2.6)–(2.8), whence it follows from (2.5) that $\|F(t, t_0, x_0)\| = 0$ for $t \geq t_0$, and this proves the invariance of M. Note that conditions $\|F(t, t_0, x_0)\| = 0$ and (2.9) are equivalent if $x_0 \in M$.

Pick any $\varepsilon_1 > 0$ $(\varepsilon_1 < H)$, and choose $\delta = b^{-1}(g(\varepsilon_1))$. If $\|y_0\| < \delta$, then from properties (2.5)–(2.8) we have

$$\begin{aligned} g(\|F(t, t_0, x_0)\|) &\leq V(t, \Phi(t, t_0, x_0)) \leq V(t_0, x_0) \\ &\leq b(\|y_0\|) < b(b^{-1}(g(\varepsilon_1))) = g(\varepsilon_1), \end{aligned}$$

whence it follows $\|F(t, t_0, x_0)\| < \varepsilon_1$ for $t > t_0$. This proves the uniform stability of M.

The uniform stability of M implies that for every positive ε_2 ($\varepsilon_2 < H$) there exists $H_0 = H_0(\varepsilon_2) > 0$ such that for any $t_0 \in \mathbb{R}_+$, $x_0 \in B_{H_0}$ the inequality $\|F(t, t_0, x_0)\| < \varepsilon_2$ holds for $t > t_0$. Consider the solution $\Phi(t, t_0, x_0)$ of Eqs. (2.1) and (2.2) with $x_0 \in B_{H_0}$. Since $x_0 \in B_{H_0}$, then $\|y_0\| \le H_0$, whence

$$V(t_0, x_0) \le b(\|y_0\|) \le b(H_0). \tag{2.10}$$

Let ε be any sufficiently small positive number. Denote $T(\varepsilon) := \frac{b(H_0)}{c(b^{-1}(g(\varepsilon)))}$. Let us show that there exists $\sigma \in [0, T]$ such that

$$V(t_0 + \sigma, \Phi(t_0 + \sigma, t_0, x_0)) < g(\varepsilon). \tag{2.11}$$

Suppose the opposite: for any $\sigma \in [0, T]$ the inequality $V(t_0 + \sigma, \Phi(t_0 + \sigma, t_0, x_0)) \ge g(\varepsilon)$ holds, whence we have

$$\|F(t, t_0, x_0)\| \ge b^{-1}(V(t, \Phi(t, t_0, x_0))) \ge b^{-1}(g(\varepsilon)) \tag{2.12}$$

for $t_0 \le t \le t_0 + T$. From inequalities (2.12) we derive that

$$\frac{\mathrm{d}V(t, \Phi(t, t_0, x_0))}{\mathrm{d}t} \le -c(b^{-1}(g(\varepsilon))), \quad t \in \mathbb{R}_+, x_0 \in B_{\varepsilon_2}, t \ne \tau_i (i \in \mathbb{N}), c \in \mathcal{K}.$$

This inequality together with (2.5) and (2.8) imply

$$0 < g(\|F(t, t_0, x_0)\|) \le V(t, \Phi(t, t_0, x_0)) \le V(t_0, x_0) - c(b^{-1}(g(\varepsilon)))(t - t_0).$$

For $t - t_0 = T$ we have $V(t_0, x_0) - b(H_0) > 0$, but this contradicts relations (2.10). This proves the existence of $\sigma \in [0, T]$ such that inequality (2.11) is valid. Since V does not increase along the solution $\Phi(t, t_0, x_0)$, then $V(t, \Phi(t, t_0, x_0)) < g(\varepsilon)$ for $t \ge t_0 + \sigma$. This implies $\|F(t, t_0, x_0)\| < \varepsilon$ for $t \ge t_0 + \sigma$. Hence M is uniformly attractive, and its domain of attraction contains the set B_{H_0}. This completes the proof. □

Definition 2.6. System (1.1) and (1.2) is said to be periodic with respect to t with the period ω if there exists a $q \in \mathbb{N}$, such that

$$J_{i+q}(x) \equiv J_i(x), \ \tau_{i+q} = \tau_i + \omega \ (i = 1, 2, \ldots); \quad X(t + \omega, x) \equiv X(t, x). \tag{2.13}$$

3. The converse theorem

Lemma 3.1. *Assume that there are p points of impulse effect in $(t_0, t_1]$, the function Y satisfies condition* (P1) *in the domain $\mathbb{R}_+ \times B_H$, and J_{yi} $(i \in \mathbb{N})$ satisfy conditions* (P2) *in B_H, and solutions $\Phi(t, t_0, x_1)$ and $\Phi(t, t_0, x_2)$, where $x_1 = (y_1, z_1)$, $y_1 = (y_1^1, \ldots, y_1^l)$, $z_1 = (z_1^1, \ldots, z_1^m)$, and $x_2 = (y_2, z_2) = (y_2^1, \ldots, y_2^l, z_2^1, \ldots, z_2^m)$, lie in B_H for $t \in (t_0, t_1]$. Then*

$$\|F(t, t_0, x_1) - F(t, t_0, x_2)\| \le (1 + L)^p \|y_1 - y_2\| \mathrm{e}^{L(t_1 - t_0)}. \tag{3.1}$$

Proof. First assume that $p = 1$, i.e. there is one point of impulse effect $t = \tau_1$ in $(t_0, t_1]$. We have

$$F^s(\tau_1^-, t_0, x_1) - F^s(\tau_1^-, t_0, x_2) = (y_1^s - y_2^s) + \int_{t_0}^{\tau_1} [Y_s(t, \Phi(t, t_0, x_1)) - Y_s(t, \Phi(t, t_0, x_2))]\mathrm{d}t, \quad s = 1, 2, \ldots, l,$$

whence using property (P1) we obtain

$$|F^s(\tau_1^-, t_0, x_1) - F^s(\tau_1^-, t_0, x_2)| \le \|y_1 - y_2\| + L \int_{t_0}^{\tau_1} \|F(t, t_0, x_1) - F(t, t_0, x_2)\|\mathrm{d}t, \quad s = 1, 2, \ldots, l,$$

$$\|F(\tau_1^-, t_0, x_1) - F(\tau_1^-, t_0, x_2)\| = \sup_{1\le s\le l} |F^s(\tau_1^-, t_0, x_1) - F^s(\tau_1^-, t_0, x_2)| \le \|y_1 - y_2\| + L \int_{t_0}^{\tau_1} \|F(t, t_0, x_1) - F(t, t_0, x_2)\|\mathrm{d}t.$$

Using the Gronwall–Bellman lemma we obtain

$$\|F(\tau_1^-, t_0, x_1) - F(\tau_1^-, t_0, x_2)\| \leq \|y_1 - y_2\|e^{L(\tau_1 - t_0)}. \tag{3.2}$$

From property (P2) and inequality (3.2) we have

$$\begin{aligned}\|F(\tau_1^+, t_0, x_1) - F(\tau_1^+, t_0, x_2)\| &\leq \|F(\tau_1^-, t_0, x_1) - F(\tau_1^-, t_0, x_2)\| \\ &\quad + \|J_{y1}(F(\tau_1^-, t_0, x_1), E(\tau_1^-, t_0, x_1)) - J_{y1}(F(\tau_1^-, t_0, x_2), E(\tau_1^-, t_0, x_2))\| \\ &\leq (1 + L)\|y_1 - y_2\|e^{L(\tau_1 - t_0)}. \end{aligned} \tag{3.3}$$

Indeed,

$$\begin{aligned}\|F(t_1, t_0, x_1) - F(t_1, t_0, x_2)\| &\leq \|F(\tau_1^+, t_0, x_1) - F(\tau_1^+, t_0, x_2)\| \\ &\quad + L\int_{\tau_1}^{t_1} \|F(t, t_0, x_1) - F(t, t_0, x_2)\|dt\end{aligned}$$

whence from (3.3) by the Gronwall–Bellman lemma we get

$$\begin{aligned}\|F(t_1, t_0, x_1) - F(t_1, t_0, x_2)\| &\leq \|F(\tau_1^+, t_0, x_1) - F(\tau_1^+, t_0, x_2)\|e^{L(t_1 - \tau_1)} \\ &\leq (1 + L)\|y_1 - y_2\|e^{L(t_1 - t_0)}.\end{aligned}$$

For $p > 1$ by means of induction it is possible to show the truth of estimate (3.1). □

Corollary 3.1. *If initial conditions $x_1 = (y_1, z_1)$ and $x_2 = (y_2, z_2)$ of two solutions $\Phi(t, t_0, x_1)$ and $\Phi(t, t_0, x_2)$ of system (1.1) and (1.2) are such that $y_1 = y_2$, then $F(t, t_0, x_1) \equiv F(t, t_0, x_2)$ for $t \geq t_0$.*

Lemma 3.2. *Suppose that $\psi(\tau) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative bounded piecewise continuous function approaching zero as $\tau \to \infty$, with discontinuity points of the first kind $\tau_1, \ldots, \tau_n, \ldots$, such that $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{i\to\infty} \tau_i = +\infty$. Suppose that $\psi(\tau_i) = \psi(\tau_i^-)$, $i \in \mathbb{N}$, and on the set $\bigcup_{i=1}^{\infty}(\tau_{i-1}, \tau_i)$ the function $\psi(\tau)$ has a derivative $\psi'(\tau)$, satisfying the inequality $|\psi'(\tau)| \leq P$. Then the function $f(t) = \sup_{t\leq\tau<\infty} \psi(\tau)$ at any value of $t \in \mathbb{R}_+$ has one-sided derivatives such that*

$$-P \leq f'(t^-) \leq 0, \qquad -P \leq f'(t^+) \leq 0. \tag{3.4}$$

Proof. Note that the curve $y = f(t)$ for $t \in \mathbb{R}_+$ consists of alternating parts of the curve $y = \psi(t)$, where $\psi(t)$ is non-increasing and segments where the function $f(t)$ is constant; that is $f(t)$ is a piecewise continuous monotonically non-increasing function approaching zero as $t \to \infty$. The discontinuity points can occur only at the points $t = \tau_i$ $(i \in \mathbb{N})$. For $t \in \mathbb{R}_+$ this function has the one-sided derivatives $f'(t^\pm) = f'(t\pm 0)$ satisfying conditions (3.4), as required. □

Lemma 3.3. *Suppose that $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$, and $f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are two bounded nonnegative piecewise continuous functions having one-sided limits at the discontinuity points and such that*

$$\lim_{t\to\infty} f_1(t) = 0, \qquad \lim_{t\to\infty} f_2(t) = 0.$$

Then

$$\left|\sup_{t\in\mathbb{R}_+} f_1(t) - \sup_{t\in\mathbb{R}_+} f_2(t)\right| \leq \sup_{t\in\mathbb{R}_+} |f_1(t) - f_2(t)|.$$

Proof. Lemma 3.3 is obviously true in the case $\sup_{t\in\mathbb{R}_+} f_1(t) = \sup_{t\in\mathbb{R}_+} f_2(t)$. Suppose that $\sup_{t\in\mathbb{R}_+} f_1(t) \neq \sup_{t\in\mathbb{R}_+} f_2(t)$. Without loss of generality we assume that $\sup_{t\in\mathbb{R}_+} f_1(t) > \sup_{t\in\mathbb{R}_+} f_2(t)$. Since the functions $f_1(t)$, $f_2(t)$, being non-negative and bounded, approach zero as $t \to \infty$, there exist finite values t_1, t_2, t_3 such that $\sup_{t\in\mathbb{R}_+} f_1(t) = f_1(t_1^\pm)$, $\sup_{t\in\mathbb{R}_+} f_2(t) = f_2(t_2^\pm)$, and $\sup_{t\in\mathbb{R}_+} |f_1(t) - f_2(t)| = |f_1(t_3^\pm) - f_2(t_3^\pm)|$. Here $f_k(t_i^\pm)$ $(k = 1, 2;\ i = 1, 2, 3)$ denotes either the value or the one-sided limit of the function f_k at the point t_i. Consequently, $f_1(t_1^\pm) - f_2(t_2^\pm) \leq f_1(t_1^\pm) - f_2(t_1^\pm) \leq |f_1(t_3^\pm) - f_2(t_3^\pm)|$, which proves Lemma 3.3. □

Theorem 3.1. *Suppose that property* (P1)–(P3) *hold, the invariant set* (2.3) *of system* (2.1) *and* (2.2) *is uniformly asymptotically stable, and its domain of attraction contains the set* B_{H_*} $(0 < H_* < H)$. *Then there exist constants* $P > 0,\ L_1 > 0$ *and functions* $g \in \mathcal{K},\ b \in \mathcal{K},\ c \in \mathcal{K},\ V : \mathbb{R}_+ \times B_{H_*} \to \mathbb{R}_+$ *such that* $V \in \mathcal{V}_0$,

$$|V(t, x_1) - V(t, x_2)| \le L_1 \|y_1 - y_2\| \tag{3.5}$$

for $t \in \mathbb{R}_+,\ x_1 = (y_1, z_1) \in B_{H_*},\ x_2 = (y_2, z_2) \in B_{H_*},\ |V(t_1, x) - V(t_2, x)| \le L_1|t_1 - t_2|$ *for* $x \in B_{H_*},\ t_1 \in (\tau_{i-1}, \tau_i),\ t_2 \in (\tau_{i-1}, \tau_i),\ (i \in \mathbb{N})$, *conditions* (2.5)–(2.8) *are satisfied, and* $D^+V(t, x) \ge -P$.

If system (2.1) *and* (2.2) *is periodic with period* ω, *then the function* V *can also be chosen to be periodic in* t *with the period* ω.

Proof. Let $\varphi(t)$ be a monotonically decreasing function satisfying the inequality

$$\|F(t, t_0, x_0)\| \le \varphi(t - t_0) \quad \text{for } t \ge t_0 \tag{3.6}$$

for any $x_0 \in B_{H_*}$, and such that $\lim_{t\to\infty} \varphi(t) = 0$. The existence of such function $\varphi(t)$ follows from the property of uniform asymptotic stability of the invariant set M in the sense of Definition 2.4. (It suffices to choose for $\varphi(t)$ any continuous positive function that is monotonically decreasing to zero and satisfies the inequality $\varphi(t) > \varepsilon$ for $t \in [\sigma(\varepsilon), \sigma(\frac{\varepsilon}{2})]$).

Let $Q(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonically increasing continuous function such that $\lim_{t\to\infty} Q(t) = +\infty$. In [33, p. 452–458] it is shown that there exists a continuously differentiable function $g = g(\varphi) : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$g \in \mathcal{K}, \qquad g' \in \mathcal{K}, \tag{3.7}$$

$$\int_0^\infty g(\varphi(\tau))\mathrm{d}\tau = N_1 < +\infty, \tag{3.8}$$

$$\int_0^\infty g'(\varphi(\tau))Q(\tau)\mathrm{d}\tau = N_2 < +\infty, \tag{3.9}$$

$$g'(\varphi(\tau))Q(\tau) < N_3 \quad \text{for all } \tau \ge 0, \tag{3.10}$$

where N_1, N_2, N_3 are positive constants.

Let us show that the function

$$V(t, x) = \int_t^\infty g(\|F(\tau, t, x)\|)\mathrm{d}\tau + \sup_{t\le\tau<\infty} g(\|F(\tau, t, x)\|) \tag{3.11}$$

where $x = (y, z)$ satisfies all the conditions of the theorem.

Integral (3.8) converges; hence by estimate (3.6) the integral in the right-hand side of (3.11) converges. Consequently, the function V is defined in the domain

$$\mathbb{R}_+ \times B_{H_*}. \tag{3.12}$$

Note that $\sup_{t\le\tau<\infty} \|F(\tau, t, x)\| \ge \|y\|$. By (3.7) we obtain

$$\int_t^\infty g(\|F(\tau, t, x)\|)\mathrm{d}\tau \ge 0, \qquad \sup_{t\le\tau<\infty} g(\|F(\tau, t, x)\|) \ge g(\|y\|),$$

that is, the function V satisfies inequality (2.5).

From estimate (3.6) we obtain that $\|F(t, t_0, x_0)\| \le \varphi(0)$ for all t_0, x_0 from domain (3.12); hence

$$V(t, x) \le \int_0^\infty g(\varphi(\tau))\mathrm{d}\tau + g(\varphi(0)) = N_4 = \text{const}.$$

Consequently, the function V is uniformly bounded in domain (3.12). Let us show that the function V satisfies inequality (3.5). Using Lemma 3.3 we obtain

$$|V(t, x_1) - V(t, x_2)| = \left| \int_t^\infty [g(\|F(\tau, t, x_1)\|) - g(\|F(\tau, t, x_2)\|)]\mathrm{d}\tau \right.$$
$$\left. + \left[\sup_{t\le\tau<\infty} g(\|F(\tau, t, x_1)\|) - \sup_{t\le\tau<\infty} g(\|F(\tau, t, x_2)\|)\right] \right|$$

$$\leq \int_t^\infty g'_\varphi(\sup(\|F(\tau,t,x_1)\|, \|F(\tau,t,x_2)\|)\|F(\tau,t,x_1) - F(\tau,t,x_2)\|)\mathrm{d}\tau + \sup_{t\leq\tau<\infty} |g(\|F(\tau,t,x_1)\|) - g(\|F(\tau,t,x_2)\|)|. \quad (3.13)$$

According to Lemma 3.1 we have

$$\|F(\tau,t,x_1) - F(\tau,t,x_2)\| < Q(\tau-t)\|y_1 - y_2\|, \quad (3.14)$$

where $Q : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing positive continuous function satisfying the inequality

$$Q(\tau-t) > (1+L)^p \mathrm{e}^{L(\tau-t)};$$

here p is the number of points τ_i in the segment $[t,\tau]$. By property (P3) such a function does exist. Taking into account inequality (3.13), applying to the second summand in the right-hand side of (3.13) the Mean Value Theorem, and using estimates (3.9) and (3.10) we obtain

$$|V(t,x_1) - V(t,x_2)| \leq \|y_1 - y_2\| \left[\int_t^\infty g'_\varphi(\sup(\|F(\tau,t,x_1)\|, \|F(\tau,t,x_2)\|))Q(\tau-t)\mathrm{d}\tau + \sup_{t\leq\tau<\infty}(g'_\varphi(\varphi(\tau-t)))Q(\tau-t)\right] \leq (N_2+N_3)\|y_1-y_2\| \quad (3.15)$$

which proves that V satisfies condition (3.5). This implies that there exists a function $b \in \mathcal{K}$ such that inequality (2.6) holds. One can choose $b(\|y\|) = (N_2+N_3)(\|y\|)$.

We now verify that $|V(t_1,x) - V(t_2,x)| \leq L_1|t_1 - t_2|$ for $x \in B_{H_*}$, $t_1 \in (\tau_{i-1},\tau_i)$, $t_2 \in (\tau_{i-1},\tau_i)$, $i \in \mathbb{N}$, where L_1 is a constant that does not depend on i. Indeed, this follows from the fact that the first summand on the right-hand side of (3.11) is a continuous function with respect to t for $t \in \mathbb{R}_+$ and a differentiable function with respect to t for $t \neq \tau_i$ with the absolute value of the derivative uniformly bounded, while the second summand is continuous with respect to t for $t \neq \tau_i$, has bounded non-positive left and right derivatives with respect to t for $t \in \mathbb{R}_+$ by Lemma 3.2, and the absolute values of these derivatives are also uniformly bounded.

We consider $D^+V(t,x)$ along solutions of system (2.1) and (2.2). We have $D^+V = D^+\overline{V}$, where $\overline{V}$ is the result of substituting an arbitrary solution $\Phi(t,t_0,x_0)$ of system (2.1) and (2.2) into the function V. But

$$\overline{V} = \int_t^\infty g(\|F(\tau,t,\Phi(t,t_0,x_0))\|)\mathrm{d}\tau + \sup_{t\leq\tau<\infty} g(\|F(\tau,t,\Phi(t,t_0,x_0))\|)$$
$$= \int_t^\infty g(\|F(\tau,t_0,x_0)\|)\mathrm{d}\tau + \sup_{t\leq\tau<\infty} g(\|F(\tau,t,\Phi(t,t_0,x_0))\|),$$

since $F(\tau,t,\Phi(t,t_0,x_0)) \equiv F(\tau,t_0,x_0)$. Hence for $t = t_0$ we obtain

$$D^+V(t,\Phi(t,t_0,x_0))\big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty g(\|F(\tau,t_0,x_0)\|)\mathrm{d}\tau\Big|_{t=t_0} + \lim_{\xi\to 0^+}\sup \frac{1}{\xi}\left(\sup_{t_0+\xi\leq\tau<\infty} g(\|F(\tau,t_0,x_0)\|) - \sup_{t_0\leq\tau<\infty} g(\|F(\tau,t_0,x_0)\|)\right).$$

The second summand in the right-hand side of the last equality is non-positive; hence

$$D^+V(t,\Phi(t,t_0,x_0))\big|_{t=t_0} \leq -g(\|F(t_0,t_0,x_0)\|) = -g(\|y_0\|),$$

that is, D^+V satisfies relation (2.7).

For $y = F(\tau_k,t_0,x_0)$ we have $y + J_{yk}(x) = F(\tau_k^+,t_0,x_0)$, whence, bearing in mind that the second summand in (3.11) is a non-increasing function, we get that inequality (2.8) holds along the solution $\Phi(t,t_0,x_0)$ of system (2.1) and (2.2).

Now suppose that system (2.1) and (2.2) is periodic with respect to t with the period ω. We shall prove that in this case the function $V(t,x)$ defined by equality (3.11) has the property $V(t+\omega,x) \equiv V(t,x)$. Indeed,

$$V(t+\omega, x) = \int_{t+\omega}^{\infty} g(\|F(\tau, t+\omega, x)\|)\mathrm{d}\tau + \sup_{t+\omega \le \tau < \infty} g(\|F(\tau, t+\omega, x)\|).$$

Introducing a new variable s by the formula $\tau = s + \omega$ we obtain

$$V(t+\omega, x) = \int_{t}^{\infty} g(\|F(s+\omega, t+\omega, x)\|)\mathrm{d}s + \sup_{t \le s < \infty} g(\|F(s+\omega, t+\omega, x)\|). \tag{3.16}$$

Using the obvious property of solutions of periodic systems

$$F(t+\omega, t_0+\omega, x_0) = F(t, t_0, x_0) \tag{3.17}$$

by equalities (3.16) and (3.17) we obtain $V(t+\omega, x) \equiv V(t, x)$, as required. The theorem is proved. □

4. Stability of invariant sets of perturbed systems

We now demonstrate one of the possible applications of Theorem 3.1. Suppose that system (2.1) and (2.2) has the uniformly asymptotically stable invariant set M described by equality (2.3). Along with system (1.1) and (1.2) we consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x) + X_*(t, x), \quad t \in \mathbb{R}_+,\ t \ne \tau_i, \tag{4.1}$$

$$x(\tau_i^+) - x(\tau_i) = J_i(x) + J_i^*(x) = I_i(x), \quad i \in \mathbb{N}, \tag{4.2}$$

where $X, X_* = (Y_*, Z_*)$, $J_i, J_i^* = (J_{yi}^*, J_{zi}^*)$ satisfy properties (P1), (P2) respectively, and system (4.1) and (4.2) has the same invariant set M as system (1.1) and (1.2). The following theorem holds.

Theorem 4.1. *If the invariant set (2.3) of system (1.1) and (1.2) is uniformly asymptotically stable,*

$$\lim_{t\to\infty} Y_*(t, x) = 0 \tag{4.3}$$

uniformly with respect to $x \in B_H$ $(0 < H < \infty)$, and the series $\sum_{i=1}^{\infty} \|J_{yi}^*(x)\|$ converges uniformly with respect to $x \in B_H$, then the invariant set M of system (4.1) and (4.2) is also uniformly asymptotically stable.

Proof. Since the invariant set M of system (1.1) and (1.2) is uniformly asymptotically stable, there exist a function $V(t, x)$ and Hahn functions g, b, c, satisfying the conditions of Theorem 3.1. Using Yoshizawa theorem [52] we first estimate $D^+V(t, x)$ along a solution $x(t)$ of system (4.1) and (4.2) for $t \ne \tau_i,\ x \in B_{H_*}$ where $H_* < H$:

$$\begin{aligned} D^+V(t, x)\big|_{(4.1),\,(4.2)} &= \lim_{\xi\to 0^+}\sup \frac{V(t+\xi, x+\xi X(t,x)+\xi X_*(t,x)) - V(t,x)}{\xi} \\ &\le \lim_{\xi\to 0^+}\sup \frac{V(t+\xi, x+\xi X(t,x)+\xi X_*(t,x)) - V(t+\xi, x+\xi X(t,x))}{\xi} \\ &\quad + \lim_{\xi\to 0^+}\sup \frac{V(t+\xi, x+\xi X(t,x)) - V(t,x)}{\xi} \\ &\le L_1\|Y_*(t,x)\| + D^+V(t,x)\big|_{(1.1),\,(1.2)}. \end{aligned} \tag{4.4}$$

In similar fashion we estimate the value of the jump of the function V along the trajectory $x(t)$ of system (4.1) and (4.2) at the instant τ_i:

$$\begin{aligned} \Delta V_i &= V(\tau_i^+, x + J_i(x) + J_i^*(x)) - V(\tau_i, x) \\ &= [V(\tau_i^+, x + J_i(x) + J_i^*(x)) - V(\tau_i^+, x + J_i(x))] + [V(\tau_i^+, x + J_i(x)) - V(\tau_i, x)] \\ &\le V(\tau_i^+, x + J_i(x) + J_i^*(x)) - V(\tau_i^+, x + J_i(x)) \le L_1\|J_{yi}^*(x)\|. \end{aligned} \tag{4.5}$$

Recall that L_1 denotes the constant for the function V appearing in inequality (3.5).

Let us show that the set M of system (4.1) and (4.2) is uniformly stable. Pick an arbitrary $\varepsilon_1 > 0$ ($\varepsilon_1 < H_*$). Let $t_1 \in \mathbb{R}_+$ be sufficiently large. Let us show that there exists $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that the solution $x(t) = x(t, t_1, x_1) = (y(t), z(t))$ of system (4.1) and (4.2) satisfies the condition $\|y(t)\| < \varepsilon_1$ for $t > t_1$ as soon as $x_1 \in B_{\delta_1}$. We set $\delta_1 = b^{-1}(\frac{1}{2}g(\varepsilon_1))$. We assume that the value t_1 satisfies the inequality $t_1 \geq T_1$ where T_1 is so large that $L_1\|Y_*(t, x)\| < \gamma_1 = \frac{1}{2}c(\delta_1)$ for $t \geq T_1,\ x \in B_{\varepsilon_1}$ and $L_1 \sum_i \|J^*_{yi}(x)\| < \frac{1}{2}g(\varepsilon)$ for $x \in B_{\varepsilon_1}$ (here the summation is extended to those values of i, where $\tau_i \geq T_1$). Since (4.3) holds uniformly with respect to $x \in B_H$, the series $\sum_{i=1}^{\infty} \|J_{yi}(x)\|$ converges uniformly with respect x, and δ_1 depends only on ε_1, one can choose T_1 to be dependent only on ε_1. Using estimates (2.5) we obtain that

$$g(\|y(t)\|) \leq V(t, x(t)) \leq V(t_1, x_1) + \int_{t_1}^{t} D^+V(s, x(s))\Big|_{(4.1),\,(4.2)} \mathrm{d}s + \sum_i \Delta V_i, \tag{4.6}$$

where the summation on the right-hand side of (4.6) is extended to those values of i, where $\tau_i \geq t_1$. Bearing in mind that $V(t_1, x_1) < \frac{1}{2}g(\varepsilon_1),\ \sum_i \Delta V_i < \frac{1}{2}g(\varepsilon_1)$ and $D^+V(t, x)\big|_{(4.1),\,(4.2)} < -\frac{1}{2}c(\delta_1)$ for $\|y\| > \delta_1$ we obtain that at an arbitrary instant $t > t_1$ either the inequality $\|y(t)\| \leq \delta_1 < \varepsilon_1$ holds or

$$g(\|y(t)\|) \leq V(t, x(t)) \leq g(\varepsilon_1) - \frac{1}{2}c(\delta_1)(t - t_1) < g(\varepsilon_1),$$

whence $\|y(t)\| \leq \varepsilon_1$. Thus, it is proved that for any $\varepsilon_1 > 0$ there exists a value $T_1 = T_1(\varepsilon_1) > 0$ such that for any $t_1 \geq T_1$ there is $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that the inequality $\|y_1\| < \delta_1$ implies that $\|y(t)\| = \|y(t, t_1, x_1)\| < \varepsilon_1$ for $t > t_1$. From Lemma 3.1 and property (P3) we deduce that there exists $\delta > 0$ such that for any $t_0 \in [0, T_1]$ and $x_0 \in B_\delta$ the solution $x(t, t_0, x_0)$ satisfies the inequality $\|y(T_1, t_0, x_0)\| < \delta_1$ and therefore the inequality $\|y(t, t_0, x_0)\| < \varepsilon_1$ for $t > t_0$. Since δ_1 and T_1 depend only on ε_1, we conclude that δ also depends only on ε_1, which proves the uniform stability of the set M of system (4.1) and (4.2).

We now show that the set M of system (4.1) and (4.2) is uniformly attracting. For that we choose an arbitrary $\varepsilon_1 > 0$ ($\varepsilon_1 < H_*$) and the correspondent value $\delta = \delta(\varepsilon_1) > 0$ in the definition of uniform stability. Let us show that for any $\varepsilon_2 > 0$ ($\varepsilon_2 < \varepsilon_1$) there exists a $\sigma = \sigma(\varepsilon_2) > 0$ such that $\|y(t, t_0, x_0)\| < \varepsilon_2$ for all $\|y_0\| < \delta,\ t_0 \in \mathbb{R}_+$ and $t \geq t_0 + \sigma$. For that we choose $\delta_2 = \delta_2(\varepsilon_2) > 0$ such that if the solution $x(t, t_0, x_0)$ of system (4.1) and (4.2), gets into B_{δ_2} at some instant, then it stays in B_{ε_2} for all t. It is possible to choose such δ_2 by the property of uniform stability of the invariant set M of system (4.1) and (4.2) proved above. So we have the inequality $\|y(t, t_0, x_0)\| < \varepsilon_1$ for $t > t_0$. We estimate the period of time during which the solution $x(t, t_0, x_0)$ can belong to the set

$$\delta_2 \leq \|y\| \leq \varepsilon_1. \tag{4.7}$$

Let t_2 denotes an instant such that $L_1\|Y_*(t, x)\| < \gamma_2 = \frac{1}{2}c(\delta_2)$ for $t \geq t_2,\ x \in B_{\varepsilon_1}$ and $L_1\|J^*_{yi}(x)\| < \frac{1}{2}g(\varepsilon_1)$ for $\tau_i \geq t_2,\ x \in B_{\varepsilon_1}$. Since (4.3) holds uniformly with respect to $x \in B_H$, the series $\sum_{k=1}^{\infty} \|J^*_{yi}(x)\|$ converges uniformly with respect to $x \in B_H$, and γ_2 depends only on ε_2, one can choose t_2 to be dependent only on ε_2. We show that the invariant set M of system (4.1) and (4.2) reaches set (4.7) no later than at $t_3 = (b(\varepsilon_1) + \frac{1}{2}g(\varepsilon_1))/\gamma_2$. For that we estimate the value $V(t, x(t))$ for $t > t_2$:

$$\begin{aligned} 0 < g(\delta_2) &\leq V(t, x(t)) \leq V(t_2, x(t_2)) - \gamma_2(t - t_2) + \sum_i \Delta V_i \\ &\leq b(\varepsilon_1) + \frac{1}{2}g(\varepsilon_1) - \gamma_2(t - t_2). \end{aligned} \tag{4.8}$$

We obtain the required estimate from inequalities (4.8). Thus, the value σ can be chosen in the form $\sigma = t_2 + t_3$, where t_2 and t_3 depend only on ε_2. This proves that the invariant set M of system (4.1) and (4.2) is uniformly attracting and its domain of attraction contains the set B_δ. The theorem is proved. □

Theorem 4.2. *Suppose that systems (1.1), (1.2) and (4.1), (4.2) are such that the property* (P1)–(P3) *hold and there exists a positive constant h such as $\tau_{i+1} - \tau_i \geq 8h$ for $i \in \mathbb{N}$. If the invariant set M of system (1.1) and (1.2) is uniformly asymptotically stable, the function $Y_*(t, x)$ satisfies the limit relation*

$$\lim_{t\to\infty} \int_t^{t+\Omega} Y_*(s, x)\mathrm{d}s = 0 \tag{4.9}$$

uniformly with respect to $\Omega > 0,\ x \in B_H$*, and the sequence* $\{J^*_{yi}(x)\}$ *satisfies the condition*

$$\lim_{i\to\infty} \|J^*_{yi}(x)\| = 0 \tag{4.10}$$

uniformly with respect to $x \in B_H$*, then the invariant set* M *of system* (4.1) *and* (4.2) *is also uniformly asymptotically stable.*

Proof. Let $\Phi(t, t_0, x_0) = (F(t, t_0, x_0), E(t, t_0, x_0))$, $F = (F^1, \dots, F^l)$, $E = (E^1, \dots, E^m)$, and $x(t) = (x^1, \dots, x^n) = x(t, t_0, x_0) = (y(t, t_0, x_0), z(t, t_0, x_0))$, $y = (y^1, \dots, y^l)$, $z = (z^1, \dots, z^m)$ denote, respectively, the solutions of systems (1.1), (1.2) and (4.1), (4.2) satisfying the identities $\Phi(t_0, t_0, x_0) = (y_0, z_0) = x_0$ and $x(t_0, t_0, x_0) = x_0$. Since the invariant set M of system (1.1) and (1.2) is uniformly asymptotically stable, there exists a function $V(t, x)$, having the properties listed in Theorem 2.1.

First we show that M is uniformly stable with respect to system (4.1) and (4.2). Let ε be an arbitrary sufficiently small positive number ($\varepsilon < H_* < H$). We set

$$\xi = \min\left\{b^{-1}\left(\frac{1}{2}g(\varepsilon)\right),\ \frac{\varepsilon}{(1+L)^3 \mathrm{e}^{16Lh}}\right\}.$$

We shall show that any solution that starts at a sufficiently late instant at a point $x_1 = (y_1, z_1)$ satisfying the condition $\|y_1\| < \xi$ does not leave B_ε for all t. Assume the opposite: suppose that there exist a solution $x(t)$ of system (4.1) and (4.2) and instants $t_1,\ t_2$ such that $0 < t_* \le t_1 < t_2$, $\|y(t_1)\| \le \xi,\ \|y(t_2^+)\| \ge \varepsilon$ and

$$\xi < \|y(t)\| \le \varepsilon \tag{4.11}$$

for $t_1 < t \le t_2$. Here t_* is some fixed sufficiently large instant. With regard to the mutual disposition of the points $t_1,\ t_2$ and the instants of impulse effect the following cases are possible:

(1) there are no instants of impulse effect in the interval (t_1, t_2);
(2) there is one point τ_k in (t_1, t_2);
(3) there are at least two points of impulse effect in (t_1, t_2).

First we consider the case, where there are no impulses in (t_1, t_2). From Lemma 3.1, by the choice of ξ we obtain that $t_2 - t_1 > 16h$. We divide the segment $[t_1, t_2]$ into r equal segments by the points $\theta_k = t_1 + k\theta$ ($k = 1, 2, \dots, r-1$), $\theta_0 = t_1,\ \theta_r = t_2$. On $(\theta_{j-1}, \theta_j]$ we set $\Phi_{\theta_{j-1}} = x_{\theta_{j-1}} = x(\theta^+_{j-1}, t_0, x_0)$. At the same time we set $\Phi_{\theta_j} = \Phi(\theta_j, \theta_{j-1}, \Phi_{\theta_{j-1}})$ for the value of the solution of system (1.1) and (1.2) with the initial data $(\theta_{j-1}, \Phi_{\theta_{j-1}})$ at the instant θ_j; we also set $x_{\theta_j} = x(\theta_j, t_0, x_0)$.

We find

$$\begin{aligned} V(t_2, x(t_2)) - V(t_1^+, x(t_1^+)) &= \sum_{j=1}^{r}[V(\theta_j, x(\theta_j)) - V(\theta^+_{j-1}), x(\theta^+_{j-1})] \\ &= R_1 + R_2 + \sum_{j=1}^{r}[V(\theta^+_{j-1}, \Phi(\theta^+_{j-1}, t_0, x_0)) - V(\theta^+_{j-1}, x(\theta^+_{j-1}))], \end{aligned}$$

where

$$R_1 = \sum_{j=1}^{r}[V(\theta_j, \Phi(\theta_j, t_0, x_0)) - V(\theta^+_{j-1}, \Phi(\theta^+_{j-1}, t_0, x_0))],$$

$$R_2 = \sum_{j=1}^{r}[V(\theta_j, x(\theta_j)) - V(\theta_j, \Phi(\theta_j, t_0, x_0))].$$

We estimate the quantities R_1 and R_2:

$$R_1 \le \sum_{j=1}^{r} \int_{\theta_{j-1}}^{\theta_j} D^+V\bigg|_{(1.1),\,(1.2)} \mathrm{d}t \le -r\theta c(\xi) = -(t_2 - t_1)c(\xi), \tag{4.12}$$

$$|R_2| \le \sum_{j=1}^{r} |V(\theta_j, x(\theta_j)) - V(\theta_j, \Phi(\theta_j, t_0, x_0))|, \tag{4.13}$$

$$|V(\theta_j, x(\theta_j)) - V(\theta_j, \Phi(\theta_j, t_0, x_0))| \le L_1 \|y(\theta_j) - F(\theta_j)\| = L_1 \max_{1\le k\le l}(|y^k(\theta_j) - F^k(\theta_j)|), \tag{4.14}$$

$$y^k(\theta_j) - F^k(\theta_j) = \int_{\theta_{j-1}}^{\theta_j} [Y^k(t, x(t)) - Y^k(t, \Phi(t, t_0, x_0))]\mathrm{d}t + \int_{\theta_{j-1}}^{\theta_j} Y_*^k(t, x(t))\mathrm{d}t,$$

$$\begin{aligned} |y^k(\theta_j) - F^k(\theta_j)| \ \le\ & \int_{\theta_{j-1}}^{\theta_j} \left|Y^k(t, x(t)) - Y^k(t, \Phi(t, t_0, x_0))\right| \mathrm{d}t \\ & + \left|\int_{\theta_{j-1}}^{\theta_j} Y_*^k(t, x(\theta_{j-1}))\mathrm{d}t\right| + \int_{\theta_{j-1}}^{\theta_j} \left|Y_*^k(t, x(t)) - Y_*^k(t, x(\theta_{j-1}))\right| \mathrm{d}t, \end{aligned} \tag{4.15}$$

$$|Y^k(t, x(t)) - Y^k(t, \Phi(t, t_0, x_0))| \le L\|y(t) - F(t)\| \le L \cdot 2L \cdot \theta = 2L^2\theta \quad \text{for } t \in (\theta_{j-1}, \theta_j],$$

$$\int_{\theta_{j-1}}^{\theta_j} \left|Y^k(t, x(t)) - Y^k(t, \Phi(t, t_0, x_0))\right| \mathrm{d}t \le 2L^2\theta^2; \tag{4.16}$$

$$\int_{\theta_{j-1}}^{\theta_j} \left|Y_*^k(t, x(t)) - Y_*^k(t, x(\theta_{j-1}))\right| \mathrm{d}t \le L \int_{\theta_{j-1}}^{\theta_j} \|y(t) - y(\theta_{j-1})\|\mathrm{d}t \le L^2\theta^2. \tag{4.17}$$

Since (4.9) holds uniformly with respect to $\Omega > 0$, $x \in B_H$, there exists a positive continuous function $u(t)$ monotonically decreasing to zero as $t \to +\infty$ and such that

$$\left|\int_t^{t+\Omega} Y_*^k(s, x)\mathrm{d}s\right| \le u(t) \quad (x \in B_H,\ \Omega > 0,\ 1 \le k \le l),$$

whence

$$\left|\int_{\theta_{j-1}}^{\theta_j} Y_*^k(s, x(\theta_{j-1}))\mathrm{d}s\right| \le u(\theta_{j-1}) \le u(t_*).$$

Using estimates (4.12)–(4.17) we obtain

$$\begin{aligned} V(t_2, x(t_2)) - V(t_1^+, x(t_1^+)) &\le -r\theta c(\xi) + rL_1(2L^2\theta^2 + L^2\theta^2 + u(t_*)) \\ &= -\frac{15}{16}(t_2 - t_1)c(\xi) + 3L_1L^2 r(\theta^2 - 2\beta\theta + \gamma), \end{aligned} \tag{4.18}$$

where

$$\beta = \frac{c(\xi)}{96L_1L^2}, \qquad \gamma = \frac{u(t_*)}{3L^2}.$$

Using (4.5) and (4.18) one can estimate the quantity $\Delta V = V(t_2^+, x(t_2^+)) - V(t_1, x(t_1))$ irrespective of whether the instants t_1 and t_2 are points of impulse effect or not:

$$\Delta V \le -\frac{15}{16}(t_2 - t_1)c(\xi) + 3L_1L^2 r(\theta^2 - 2\beta\theta + \gamma) + L_1(\|J_{yi}^*(x(t_1))\| + \|J_{yi}^*(x(t_2))\|).$$

We assume τ_* to be so large that

$$L_1\|J_{yi}^*(x)\| < \theta c(\xi) < \frac{1}{16}(t_2 - t_1)c(\xi) \quad \text{for } \tau_i \ge \tau_*,\ x \in B_\varepsilon. \tag{4.19}$$

Since (4.10) holds, such τ_* exists.

We choose θ to satisfy the quadratic inequality

$$\theta^2 - 2\beta\theta + \gamma < 0. \tag{4.20}$$

This inequality is valid if $\theta \in (\theta^{(1)}, \theta^{(2)})$, where $\theta^{(1)} = \beta - \sqrt{\beta^2 - \gamma}$ and $\theta^{(2)} = \beta + \sqrt{\beta^2 - \gamma}$. We assume t_* to be so large that

$$u(t_*) < \frac{c^2(\xi)}{3072 L_1^2 L^2}. \tag{4.21}$$

This ensures that $\beta^2 - \gamma > 0$ and $\theta^{(2)} > \theta^{(1)} > 0$. Let us show that there exist a positive integer r and $\theta \in (\theta^{(1)}, \theta^{(2)})$ such that

$$r\theta = t_2 - t_1. \tag{4.22}$$

The inequalities $\theta^{(1)} < \theta < \theta^{(2)}$ under condition (4.22) can be written in the form

$$\theta^{(1)} < \frac{t_2 - t_1}{r} < \theta^{(2)},$$

or

$$\frac{t_2 - t_1}{\theta^{(1)}} > r > \frac{t_2 - t_1}{\theta^{(2)}}. \tag{4.23}$$

For the existence of a positive integer r satisfying inequalities (4.23) it is sufficient that

$$\frac{t_2 - t_1}{\theta^{(1)}} - \frac{t_2 - t_1}{\theta^{(2)}} > 1,$$

which holds if

$$h\left(\frac{1}{\theta^{(1)}} - \frac{1}{\theta^{(2)}}\right) > 1. \tag{4.24}$$

It is easy to verify that inequality (4.24) holds for $0 < \gamma < -2h^2 + 2\sqrt{h^4 + \beta^2 h^2}$, that is, for

$$t_* > u^{-1}\left(3L^2\left(-2h^2 + 2h\sqrt{h^2 + \beta^2}\right)\right). \tag{4.25}$$

Thus, for any

$$t_* > \max\left\{\tau_*, \frac{c^2(\xi)}{3072 L_1^2 L^2}, u^{-1}\left(3L^2\left(-2h^2 + 2h\sqrt{h^2 + \beta^2}\right)\right)\right\}$$

inequalities (4.19), (4.21) and (4.25) are satisfied, and we have

$$V(t_2^+, x(t_2^+)) - V(t_1, x(t_1)) \le -\frac{13}{16}(t_2 - t_1)c(\xi). \tag{4.26}$$

We now consider the second case where on interval (t_1, t_2) there is one point τ_k of impulse effect: $t_1 < \tau_k < t_2$. We assume that $t_* > \tau_*$. In this case simirlarly to (4.18) and (4.19) we obtain

$$\begin{aligned} V(t_2^+, x(t_2^+)) - V(t_1, x(t_1)) \le{}& [V(t_2, x(t_2)) - V(\tau_k^+, x(\tau_k^+))] \\ &+ [V(\tau_k, x(\tau_k)) - V(t_1^+, x(t_1^+))] + 3hc(\xi) \end{aligned}$$

irrespective of whether t_1 and t_2 are points of impulse effect or not. In this case the inequality $t_2 - t_1 > 16h$ also holds; hence the length of at least one of the intervals (t_1, τ_k), (τ_k, t_2) is greater than $8h$. Without loss of generality we assume that $t_2 - \tau_k \ge \tau_k - t_1$, that is, $t_2 - \tau_k \ge \frac{1}{2}(t_2 - t_1) > 8h$. Similarly to (4.26) one can show that

$$V(t_2^+, x(t_2^+)) - V(\tau_k, x(\tau_k)) \le -\frac{15}{16}(t_2 - \tau_k)c(\xi) + 2hc(\xi) < -\frac{5}{16}(t_2 - t_1)c(\xi) \tag{4.27}$$

for t_* satisfying conditions $t_* > \tau_*$ and (4.25).

The right-hand side of system (4.1) and (4.2) satisfies property (P1). Consequently, by Theorem 3.1 there exists $L_2 > 0$ such that

$$|V(\tau_k, x(\tau_k)) - V(t_1^+, x(t_1^+))| \le L_2(\tau_k - t_1), \tag{4.28}$$

where the constant L_2 does not depend on the solution, but depends only on the constants L, L_1. In the case $\tau_k - t_1 < hc(\xi)/L_2$ by (4.19), (4.25), (4.27) and (4.28) we have

$$\begin{aligned} V(t_2^+, x(t_2^+)) - V(t_1, x(t_1)) &\le [V(t_2^+, x(t_2^+)) - V(\tau_k, x(\tau_k))] + [V(\tau_k, x(\tau_k)) - V(t_1^+, x(t_1^+))] + hc(\xi) \\ &\le -\frac{5}{16}(t_2 - t_1)c(\xi) + 2hc(\xi) < -\frac{1}{8}(t_2 - t_1)c(\xi). \end{aligned} \tag{4.29}$$

If, however, $\tau_k - t_1 \ge hc(\xi)/L_2$, then dividing the interval $(t_1, \tau_k]$ into r_* equal segments of length θ_* one can show similarly to (4.26) that for

$$t_* > u^{-1}\left(3L^2\left(-2(hc(\xi)/L_2)^2 + 2hc(\xi)\Big/L_2\sqrt{(hc(\xi)/L_2)^2 + \beta^2}\right)\right) \tag{4.30}$$

we have the inequality $V(\tau_k, x(\tau_k)) - V(t_1^+, x(t_1^+)) < 0$. Consequently, in this case estimate (4.29) is also true.

We now consider the third case, where on (t_1, t_2) there are d points of impulse effect $\tau_k, \tau_{k+1}, \dots, \tau_{k+d-1}$. In the case when $\tau_k - t_1 \ge h, t_2 - \tau_{k+d-1} \ge h$, dividing each of the intervals $(t_1, \tau_k], (\tau_k, \tau_{k+1}], \dots, (\tau_{k+d-1}, t_2]$ into r segments of length θ (here the values of r and θ are, generally speaking, different for each of the segments) one can show similarly to the first case that for t_*, satisfying conditions $t_* > \tau_*$ and (4.25), estimate (4.26) holds. If, however, one of the values $(\tau_k - t_1)$, $(t_2 - \tau_{k+d-1})$ or both are less than h, then one can apply the arguments similar to those used in the first and second cases to show that for $t_* = t_*(\varepsilon)$ satisfying conditions (4.19), (4.25) and (4.30) estimate (4.29) is also true.

From inequalities (2.5) and (2.6) we have

$$\inf_{\|y\|\ge\varepsilon} V(t, x) \ge g(\varepsilon), \qquad \sup_{\|y\|\le\xi} V(t, x) \le b(\xi) \le \frac{1}{2}g(\varepsilon). \tag{4.31}$$

On the other hand, for t_* satisfying relations (4.19), (4.25) and (4.30) under the assumption that there exist instants $t_2 > t_1 \ge t_*$ such that $y(t_1) \le \xi$ and $y(t_2^+) \ge \varepsilon$ we have

$$V(t_2^+, x(t_2^+)) - V(t_1, x(t_1)) \le -\frac{1}{8}(t_2 - t_1)c(\xi). \tag{4.32}$$

Inequalities (4.32) contradict relations (4.31). The contradiction thus obtained shows that for any system of the form (4.1) and (4.2) there exist no instants t_1, t_2 such that $t_2 > t_1 \ge t_*(\varepsilon)$, $\|y(t_1)\| \le \xi$, $\|y(t_2)\| \ge \varepsilon$. By Lemma 3.1, there exists $\delta > 0$ such that $\|y(t_*, t_0, x_0)\| < \xi$ for any $t_0 \in [0, t_*]$, $x_0 \in B_\delta$. Since ξ and t_* depend only on ε, $\delta > 0$ also depends only on ε. This proves the uniform stability of the invariant set M of system (4.1) and (4.2).

We now show that the invariant set M of system (4.1) and (4.2) is uniformly asymptotically stable. Let λ be some fixed number such that $(0 < \lambda < H_*)$. We proved the uniform stability of the invariant set M of system (4.1) and (4.2), so there exists an $\eta = \eta(\lambda) > 0$ such that any solution $x(t) = x(t, t_0, x_0)$ of system (4.1) and (4.2) satisfying the condition $x_0 \in B_\eta$ satisfies the condition $x(t) \in B_\lambda$ for any $t > t_0 \ge 0$. We shall show that for any $\rho > 0$ $(\rho < \lambda)$ one can find a $\sigma = \sigma(\rho) > 0$ such that $\|y(t)\| < \rho$ for arbitrary $x_0 \in B_\eta$, $t_0 \in \mathbb{R}_+$, $t \ge t_0 + \sigma$.

Suppose that $0 < \rho < \lambda$. According to the uniform stability of the invariant set M of system (4.1) and (4.2), there exists a $\delta = \delta(\rho) > 0$ such that the condition $x(T_0) \in B_\delta$ implies that $x(t) \in B_\rho$ for any $t \ge T_0 \ge 0$. We estimate the time during which the trajectory can be in the domain $B_\lambda \setminus B_\delta$. Similarly to (4.32) one can show that

$$V(t, x(t)) - V(T_1, x(T_1)) < -\frac{1}{8}(t - T_1)c(\delta) \tag{4.33}$$

for $t \ge T_1$ where T_1 depends only on $\delta(\rho)$, that is $T_1 = T_1(\rho)$. Then inequality (4.33) implies that

$$t - T_1 < \frac{8[V(T_1, x(T_1)) - V(t, x(t))]}{c(\delta)} \le \frac{8[b(\lambda) - g(\delta)]}{c(\delta)} = T_2(\rho).$$

Setting $\sigma(\rho) = T_1(\rho) + T_2(\rho)$ we obtain that $\|y(t, t_0, x_0)\| < \rho$ for any $t_0 \in \mathbb{R}_+$, $x_0 \in B_\eta$, $t \ge t_0 + \sigma$. This means precisely that the integral set M of system (4.1) and (4.2) is uniformly asymptotically stable and its domain of attraction contains the set B_η. $\square$

5. On the stability of invariant sets of periodic systems

Further, we assume that system (1.1) and (1.2) is periodic in t with the period to ω.

Theorem 5.1. *If the invariant set M of system (1.1) and (1.2) is stable, then it is uniformly stable.*

Proof. Conditions (2.13) imply that

$$\Phi(t+\omega, t_0+\omega, x_0) \equiv \Phi(t, t_0, x_0), \tag{5.1}$$

hence it suffices to prove that for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|F(t, t_0, x_0)\| \le \varepsilon$ holds for $t \ge t_0$ and for all $t_0 \in [0, \omega)$, $x_0 \in B_\delta$. By assumption, for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $x_\omega = \Phi(\omega, t_0, x_0)$ satisfies the condition $x_\omega \in B_{\delta_1}$, then $\Phi(t, \omega, x_\omega) \in B_\varepsilon$ for $t \ge \omega$. If the condition $x_0 = \Phi(t_0, t_0, x_0) \in B_\delta$ holds at any time $t_0 \in [0, \omega)$, then Lemma 3.1 implies that if one choose $\delta = (1+L)^{-q} e^{-L\omega}\delta_1$, then $\Phi(t, t_0, x_0) \in B_\varepsilon$. This completes the proof of the theorem. □

Theorem 5.2. *If the invariant set M of Eqs. (1.1) and (1.2) is asymptotically stable, then it is uniformly asymptotically stable.*

Proof. By the assumptions, M is asymptotically stable; hence (2.4) holds in the domain

$$t_0 \in \mathbb{R}_+, \qquad x_0 \in B_\lambda, \tag{5.2}$$

where λ is a sufficiently small positive number. Now, let us prove that this limit relation holds uniformly in t_0, x_0 from (5.2), i.e. for every $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that the inequality $\|F(t, t_0, x_0)\| \le \varepsilon$ holds for all $t \ge t_0 + \sigma$. Since the system is periodic, we assume that t_0 belongs to the segment $[0, \omega]$. First, let us define the number $\eta = \eta(\varepsilon) > 0$ from the condition

$$\|F(t, t_0, x_0)\| \le \varepsilon \quad \text{for } x_0 \in B_\eta, t > t_0. \tag{5.3}$$

This is possible because of the uniform stability of M. Arguing by contradiction, assume that the number $\sigma = \sigma(\varepsilon)$ does not exist. Then, for an arbitrary large integer m, there exists a $t_m > m\omega$, initial values $t_{0m} \in [0, \omega]$ and $x_{0m} = (y_{0m}, z_{0m}) \in B_\lambda$ such that

$$\|F(t_m, t_{0m}, x_{0m})\| > \varepsilon. \tag{5.4}$$

Since the sequence of points $\{t_{0m} \times y_{0m}\}$ belongs to a compact set, one can choose a subsequence of this sequence which converges to some point $t_* \times y_*$ where $t_* \in [0, \omega]$, $\|y_*\| \le \lambda$. Hence (2.4) holds for the initial values $t_0 = t_*$, $x_0 = x_* = (y_*, z_*)$ where $z_* \in \mathbb{R}^m$ is arbitrary. Then there exists large enough $k = k(\varepsilon)$ such that

$$\|F(t_* + k\omega, t_*, x_*)\| < \frac{1}{2}\eta(\varepsilon). \tag{5.5}$$

Corollary 3.1 implies

$$\|F(t_* + k\omega, t_*, x_{0m})\| \equiv \|F(t_* + k\omega, t_*, x_*)\| < \frac{1}{2}\eta(\varepsilon), \quad m \in \mathbb{N}.$$

Denote $x^{(k)} = (y^{(k)}, z^{(k)})$ where $x^{(k)} = \Phi(t_{0m} + k\omega, t_{0m}, x_{0m})$. Since $t_{0m} \to t_*$, $x_{0m} \to x_*$, there exist an arbitrary large values of m for which we have the inequality

$$\|y^{(k)}\| < \eta(\varepsilon), \tag{5.6}$$

where $y^{(k)} = F(t_{0m} + k\omega, t_{0m}, x_{0m})$. Estimates (5.6) and (5.3)imply that for all $t > t_{0m}$ we have $\|F(t, t_{0m}, x^{(k)})\| \le \varepsilon$, hence identity (5.1) and the uniqueness property of the solution imply the estimate

$$\varepsilon \ge \|F(t, t_{0m}, x^{(k)})\| \equiv \|F(t + k\omega, t_{0m} + k\omega, x^{(k)})\| \equiv \|F(t + k\omega, t_{0m}, x_{0m})\|.$$

The obtained inequality contradicts assumption (5.4), because there exists an instant t_m, such that $t_m > k\omega$. The contradiction proves that (2.4) is uniform in t_0 and x_0. This completes the proof of the theorem. □

Next, we apply the ideas of Barbashin–Krasovskii to the stability analysis of the invariant set of a system of impulsive differential equations by using Lyapunov's second method.

Let us introduce the following definition.

Definition 5.1. We say that the function $g : \mathbb{R}_+ \to \mathbb{R}^s,\ s \in \mathbb{N}$ is not eventually vanishing if for any $M > 0$ there exists a $t > M$ such that $g(t) \neq 0$. We say that the sequence of numbers $\{u_k\}_{k=1}^{\infty}$ is not eventually vanishing if for any positive integer M there exists a $k > M$ such that $u_k \neq 0$.

Theorem 5.3. *Suppose that every solution of system (1.1) and (1.2) with initial data $(t_0, x_0) \in \mathbb{R}_+ \times B_H$ is z- bounded, there exists a function $V(t, x) \in \mathcal{V}_1$ which is periodic in t with period ω and satisfies the conditions*

$$a(\|y\|) \leq V(t, x) \leq b(\|y\|), \quad a \in \mathcal{K},\ b \in \mathcal{K}, \tag{5.7}$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0 \quad \text{for } (t, x) \in G,$$
$$\Delta V_i(x) = V(\tau_i^+, x + J_i(x)) - V(\tau_i, x) \leq 0, \quad i \in \mathbb{N}.$$

If along any eventually vanishing solution of Eqs. (1.1) and (1.2), at least one of the following conditions holds:

(i) *the function* $\mathrm{d}V/\mathrm{d}t$ *is not eventually vanishing,*
(ii) *the sequence* $\{\Delta V_i\}$ *is not eventually vanishing,*

then the invariant set M of system (1.1) and (1.2) is uniformly asymptotically stable.

Proof. The stability property of M can be proved just as in Theorem 2.1. Theorem 5.1 implies that M is uniformly stable, i.e. for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in B_\delta$ the inequality $\|F(t, t_0, x_0)\| \leq \varepsilon$ holds for all $t > t_0$. Let us prove that any trajectory $\Phi(t, t_0, x_0)$ with such initial conditions possesses property (2.4).

Consider the function $v(t) = V(t, \Phi(t, t_0, x_0))$. It is not increasing and is bounded below; hence the limit

$$\lim_{t\to\infty} v(t) = \eta \geq 0$$

exists. Let us prove that $\eta = 0$. Suppose that the converse is true:

$$\eta = \lim_{t\to\infty} V(t, \Phi(t, t_0, x_0)) > 0. \tag{5.8}$$

Consider the sequence of points $\{x_k\} = \{(y_k, z_k)\}$ where $x_k = \Phi(t_0 + k\omega, t_0, x_0)$. Taking into account the fact that $\|y_k\| \leq \varepsilon < H$, and $\{z_k\}$ is bounded, one can conclude that there exists a subsequence converging to the point $x_* \in B_\varepsilon$. Without loss of generality, we shall assume that the sequence $\{x_k\}$ itself converges to the point $x_* \neq 0$. Since the function V is continuous in x and periodic in t, the equality $V(t_0, x_*) = \eta$ must be satisfied. Consider the semitrajectory $\Phi(t, t_0, x_*)$ for $t \geq t_0$ and the function $V_*(t) = V(t, \Phi(t, t_0, x_*))$ along the trajectory. By the assumptions of the theorem, the function $V_*(t)$ is not increasing; moreover, there exists either an instant t_1 where the trajectory is continuous such that

$$\mathrm{d}V(t_1, \Phi(t_1, t_0, x_*))/\mathrm{d}t < 0$$

or a discontinuity point τ_s such that

$$V(\tau_s^+, \Phi(\tau_s^+, t_0, x_*)) - V(\tau_s, \Phi(\tau_s, t_0, x_*)) < 0.$$

This means that there exists an instant $t_* > t_0$ such that

$$V(t_*, \Phi(t_*, t_0, x_*)) = \eta_1 < \eta.$$

Since the sequence $\{x_k\}$ converges to the point x_*, by Lemma 3.1 the following inequality holds:

$$\|\Phi(t_*, t_0, x_*) - \Phi(t_*, t_0, x_k)\| < \gamma$$

for all $k > k_0(\gamma)$, for an arbitrary number $\gamma > 0$. Hence

$$\lim_{k\to\infty} V(t_*, \Phi(t_*, t_0, x_k)) = \eta_1. \tag{5.9}$$

Taking into account the periodicity of system (1.1) and (1.2), we can write

$$\Phi(t_*, t_0, x_k) = \Phi(t_*, t_0, \Phi(t_0 + k\omega, t_0, x_0)) = \Phi(t_* + k\omega, t_0, x_0). \tag{5.10}$$

Indeed, the trajectories of system (1.1) and (1.2) starting at time t_0 and $t_0 + k\omega$ at the point x_k, respectively, will move to the points $\Phi(t_*, t_0, x_k)$ and $\Phi(t_* + k\omega, t_0, x_0)$ respectively during time $\Delta t = t_* - t_0$; this proves relation (5.11). The periodicity of the function $V(t, x)$ in t yields the equality $V(t_*, x) = V(t_* + k\omega, x)$. Hence taking into account (5.10), condition (5.9) can be rewritten as follows:

$$\lim_{k\to\infty} V(t_* + k\omega, \Phi(t_* + k\omega, t_0, x_0)) = \eta_1. \tag{5.11}$$

But since $\eta_1 < \eta$, relation (5.11) contradicts the inequality $V(t, x(t, t_0, x_0)) \geq \eta$. This contradiction proves that assumption (5.8) was incorrect, i.e. $\eta = 0$. Condition (5.7) justifies (2.4), which proves the asymptotic stability of the trivial solution. By using Theorem 5.2, we conclude that the invariant set M of system (1.1) and (1.2) is uniformly asymptotically stable. □

Theorem 5.4. *Suppose that every solution of system (1.1) and (1.2) with initial data $(t_0, x_0) \in \mathbb{R}_+ \times B_H$ is z- bounded, there exists a function $V(t, x)$ which is periodic in t with the period ω, continuously differentiable on the domain G, and satisfies the conditions*

$$|V(t, x)| \leq b(\|y\|), \quad b \in \mathcal{K}; \tag{5.12}$$

$$\frac{dV}{dt} \geq 0 \quad \text{for } (t, x) \in G, \tag{5.13}$$

$$\Delta V_i(x) = V(\tau_i^+, x + I_i(x)) - V(\tau_i, x) \geq 0. \tag{5.14}$$

Moreover, suppose that, along any eventually vanishing solution of Eqs. (1.1) and (1.2), at least one of the following conditions holds:

(i) *the function* dV/dt *is not eventually vanishing,*
(ii) *the sequence* $\{\Delta V_i\}$ *is not eventually vanishing.*

If any arbitrary small neighborhood of the origin for any $t > 0$ contains a point x such that $V(t, x) > 0$, then the invariant set M of system (1.1) and (1.2) is unstable.

Proof. Suppose that $\varepsilon < H$ is a positive number. Let us choose an arbitrary $t_0 \in \mathbb{R}_+$ and an arbitrary small $\delta > 0$. We shall prove that there exist $x_0 \in B_\delta$ and $t > t_0$ such that $\|F(t, t_0, x_0)\| > \varepsilon$. To this end, take $x_0 \in B_\delta$ so that $V(t_0, x_0) = V_0 > 0$. Suppose the converse:

$$\|F(t, t_0, x_0)\| \leq \varepsilon \tag{5.15}$$

for all $t > t_0$. Condition (5.12) implies

$$|V(t, x)| < V_0 \quad \text{for } \|y\| < b^{-1}(V_0) = \eta, \quad t \in \mathbb{R}_+.$$

Taking into account assumptions (5.13)–(5.15), we conclude that the semitrajectory $\Phi(t, t_0, x_0)$ satisfies the conditions

$$\eta \leq \|F(t, t_0, x_0)\| \leq \varepsilon.$$

Consider the sequence of points $\{x_j\}$ given by $x_j = \Phi(t_0 + j\omega, t_0, x_0)$ $(j = 1, 2, \ldots)$. Taking into account the fact that this sequence belongs to a compact set, we can choose a subsequence converging to the point $x_* = (y_*, z_*)$ satisfying conditions $\eta \leq \|y_*\| \leq \varepsilon$. Without loss of generality, we assume that the sequence $\{x_j\}$ converges to the point x_*.

The function $v(t) = V(t, \Phi(t, t_0, x_0))$ is monotone nondecreasing and bounded above by a constant $b(\varepsilon)$; hence the limit

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} V(t, \Phi(t, t_0, x_0)) = v_0 = V(t_0, x_*)$$

exists and we have

$$V(t, \Phi(t, t_0, x_0)) \leq v_0. \tag{5.16}$$

Now, let us consider the semitrajectory $\Phi(t, t_0, x_*)$ and $t > t_0$. By the assumption of the theorem, there exists a point t_1 such that

$$\mathrm{d}V(t_1, \Phi(t_1, t_0, x_*))/\mathrm{d}t > 0$$

or a point τ_s such that

$$\Delta V_s = V(\tau_s^+, \Phi(\tau_s, t_0, x_*)) + J_s(\Phi(\tau_s, t_0, x_*)) - V(\tau_s, \Phi(\tau_s, t_0, x_*)) > 0.$$

This means that there exists an instant $t_* > t_0$ such that

$$V(t_*, \Phi(t_*, t_0, x_*)) = v_1 > v_0.$$

Since the sequence $\{x_j\}$ converges to the point x_*, Lemma 3.1 implies the inequality

$$\|x(t_*, t_0, x_*) - x(t_*, t_0, x_j)\| < \gamma$$

for all $j > N(\gamma)$, for an arbitrary constant $\gamma > 0$. Hence

$$\lim_{j\to\infty} V(t_*, \Phi(t_*, t_0, x_j)) = v_1. \tag{5.17}$$

Taking into account the periodicity of system (1.1) and (1.2), we can write

$$\Phi(t_*, t_0, x_j) = \Phi(t_* + j\omega, t_0, x_0). \tag{5.18}$$

The periodicity in t of the function $V(t, x)$ yields the equality $V(t_*, x) = V(t_* + j\omega, x)$, hence, by taking (5.18) into account, condition (5.17) can be rewritten as

$$\lim_{j\to\infty} V(t_* + j\omega, \Phi(t_* + j\omega, t_0, x_0)) = v_1. \tag{5.19}$$

On the other hand, relation (5.19) contradicts inequality (5.16), because $v_1 > v_0$. This contradiction proves that asuumption (5.15) was incorrect, i.e. the invariant set M of system (1.1) and (1.2) is unstable; this proves the theorem. □

Acknowledgment

Research for this article was supported by Central European University ("CEU") Special and Extension Programs. The opinions expressed herein are the author's own and do not necessarily express the views of CEU.

References

[1] J. Angelova, A. Dishliev, Optimization problems for one-impulsive models from population dynamics, Nonlinear Anal. 39 (2000) 483–497.
[2] J. Angelova, A. Dishliev, S. Nenov, Comparison of zero-solutions of systems ode via asymptotical stability, Nonlinear Anal. 42 (2000) 339–350.
[3] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effect: Stability, Theory and Applications, Halsted Press, New York, Chichester, Brisbane, Toronto, 1989.
[4] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific, Singapore, 1995.
[5] S.P. Bhat, D.S. Bernstein, Geometric homogeneity with applications to finite-time stability, Math. Control Signals Systems 17 (2) (2005) 101–127.
[6] S.D. Borysenko, M. Ciarletta, G. Iovane, Integro-sum inequalities and motion stability of systems with impulse perturbations, Nonlinear Anal. 62 (2005) 417–428.
[7] T.A. Burton, Uniform asymptotic stability in functional differential equations, Proc. Amer. Math. Soc. 68 (1978) 195–199.
[8] T.A. Burton, L. Hatvani, On nonuniform asymptotic stability for nonautonomous functional-differential equations, Differential Integral Equations 3 (1974) 285–293.
[9] T.A. Burton, L. Hatvani, Stability theorems for nonautonomous functional-differential equations by Liapunov fuctionals, Tõhoku Math. J. 41 (1989) 65–104.
[10] A. Cabada, E. Liz, Discontinuous impulsive differential equations with nonlinear boundary conditions, Nonlinear Anal. 28 (1997) 1491–1497.
[11] V. Chellaboina, S.P. Bhat, W.M. Haddad, An invariance principle for nonlinear hybrid and impulsive dynamical systems, Nonlinear Anal. 53 (2003) 527–550.
[12] C. Corduneanu, Some problems concerning partial stability, in: Symp. Math. Meccanica Nonlineare. Stability, 23–26 Febbrario, 1970, vol. 6, 1st ed., Academic Press, New York, 1971, pp. 141–154.
[13] C. Corduneanu, On partial stability for delay systems, Ann. Polon. Math. 29 (1975) 357–362.

[14] A.K. Gelig, A.N. Churilov, Stability and Oscillations of Nonlinear Pulse-Modulated Systems, Birkhauser, Boston, 1998.
[15] Z.-H. Guan, J. Lam, G. Chen, On impulsive autoassociative neural networks, Neural Netw. 13 (2000) 63–69.
[16] S.I. Gurgula, N.A. Perestyuk, On stability of the equilibrium in impulse systems, Mat. Fiz. 31 (1982) 9–14.
[17] W.M. Haddad, V. Chellaboina, N.A. Kablar, Non-linear impulsive dynamical systems. Part I: Stability and dissipativity, Internat. J. Control 74 (17) (2001) 1631–1658.
[18] W.M. Haddad, V. Chellaboina, S.G. Nersesov, Hybrid nonnegative and compartmental dynamical systems, Math. Prob. Eng. 8 (6) (2002) 493–515.
[19] W.M. Haddad, V. Chellaboina, S.G. Nersesov, Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control, Princeton University Press, Princeton, 2006.
[20] L. Hatvani, On the asymptotic stability of nonautonomous functional differential equations by Lyapunov functionals, Trans. Amer. Math. Soc. 354 (2002) 3555–3571.
[21] S. Hu, V. Lakshmikantham, Periodic boundary value problems for second order impulsive differential systems, Nonlinear Anal. 13 (1989) 75–87.
[22] A.O. Ignat'ev, O.A. Ignat'ev, A.A. Soliman, Asymptotic stability and instability of the solutions of systems with impulse action, Math. Notes 80 (4) (2006) 491–499.
[23] A.O. Ignatyev, Partial stability under persistent disturbances, Izv. Vyssh. Uchebn. Zaved. Mat. 2 (345) (1991) 55–60.
[24] A.O. Ignatyev, Application of the direct method of Lyapunov to the analysis of integral sets, Ukrainian Math. J. 44 (1992) 1229–1235.
[25] A.O. Ignatyev, On the existence of Lyapunov functions in problems of stability of integral sets, Ukrainian Math. J. 45 (1993) 1031–1041.
[26] S.K. Kaul, X. Liu, Generalized variation of parameters and stability of impulsive systems, Nonlinear Anal. 40 (2000) 295–307.
[27] K.K. Kenzhebaev, A.N. Stanzhitskii, Invariant sets of impulsive systems and their stability, Nonlinear Oscil. 7 (2004) 78–82.
[28] C. Kou, S. Zhang, S. Wu, Stability analysis in terms of two measures for impulsive differential equations, J. London Math. Soc. 66 (2) (2002) 142–152.
[29] G.K. Kulev, D.D. Bainov, Global stability of sets for impulsive differential systems by Lyapunov's direct method, Comput. Math. Appl. 19 (2) (1990) 17–28.
[30] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapure, New Jersey, London, 1989.
[31] V. Lakshmikantham, S. Leela, S. Kaul, Comparison principle for impulsive differential equations with variable times and stability theory, Nonlinear Anal. TMA 22 (1994) 499–503.
[32] V. Lakshmikantham, X. Liu, On quasi stability for impulsive differential systems, Nonlinear Anal. TMA 13 (1989) 819–829.
[33] I.G. Malkin, Stability of Motion, Nauka, Moscow, 1966.
[34] G. Matarazzo, N. Perestyuk, O. Chernikova, Stability of sets for impulsive differential systems, Miskolc Math. Notes 5 (2) (2004) 193–201.
[35] A.N. Michel, A.P. Molchanov, Y. Sun, Partial stability and boundedness of discontinuous dynamical systems, Nonlinear Stud. 9 (3) (2002) 225–247.
[36] A.N. Michel, A.P. Molchanov, Y. Sun, Partial stability and boundedness of general dynamical systems on metric spaces, Nonlinear Anal. 52 (2003) 1295–1316.
[37] S.I. Nenov, Impulsive controllability and optimization problems in population dynamics, Nonlinear Anal. 36 (1999) 881–890.
[38] K. Peiffer, N. Rouche, Liapunov's second method applied to partial stability, J. Mecanique 6 (1969) 20–29.
[39] N.A. Perestyuk, O.S. Chernikova, On the stability of integral sets of impulsive differential systems, Miskolc Math. Notes 2 (1) (2001) 49–60.
[40] N.A. Perestyuk, O.S. Chernikova, On the stability of invariant sets of discontinuous dynamical systems, Ukrainian Math. J. 53 (2001) 91–98.
[41] N.A. Perestyuk, O.S. Chernikova, On stability of integral sets of impulsive differential systems, Ukrainian Math. J. 54 (2002) 306–315.
[42] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Springer, New York, 1977.
[43] V.V. Rumyantsev, A.S. Oziraner, Stability and Stabilization with Respect to Part of Variables, Nauka, Moscow, 1987.
[44] A.M. Samoilenko, Elements of the Mathematical Theory of Multifrequency Vibrations, Nauka, Moscow, 1987.
[45] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapure, New Jersey, London, 1995.
[46] A.Ya. Savchenko, A.O. Ignatyev, Some Stability Problems of Nonautonomous Dynamic Systems, Naukova Dumka, Kiev, 1989.
[47] P.S. Simeonov, D.D. Bainov, Stability with respect to part of the variables in systems with impulse effect, J. Math. Anal. Appl. 117 (1) (1986) 247–263.
[48] R.J. Smith, L.M. Wahl, Distinct effects of protease and reverse transcriptase inhibition in an immunological model of HIV-1 infection with impulsive drug effects, Bull. Math. Biol. 66 (2004) 1259–1283.
[49] A.A. Soliman, On stability of perturbed impulsive differential systems, Appl. Math. Comput. 133 (2002) 105–117.
[50] V.I. Vorotnikov, Partial Stability and Control, Birkhauser, New York, 1998.
[51] Z. Wang, J. Sun, On impulsive dynamic systems, Impuls. Dyn. Syst. Appl. 3 (2005) 114–119.
[52] T. Yoshizawa, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer, New York, 1975.
[53] X. Zhang, Z. Shuai, K. Wang, Optimal impulsive harvesting policy for single population, Nonlinear Anal. RWA 4 (2003) 639–651.
[54] Y. Zhang, B. Liu, L. Chen, Extinction and permanence of a two-prey one-predator system with impulsive effect, Math. Med. Biol. 20 (2003) 309–325.