OPTIMAL CONTROL IN PARABOLIC SINGULAR PERTURBATED PROBLEM WITH OBSTACLE.

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1. *OPTIMALITY CONDITIONS.*

Consider such optimal control problem with an obstacle: to find $u(t) \in U = \{v : v(t) \in L_2(0,T), \mid v(t) \mid \leq \xi$ for a.e. $t \in [0,T]\}$ such that

$$I(v) = \frac{1}{2} \int_0^T \Big(\int_\Omega (y(x,t) - z(x))^2 dx + \nu v^2(t) \Big) dt \ \rightarrow \ min, \tag{1}$$

where $y(x,t)$ is the solution of variational inequality of parabolic type in [1-2]

$$\begin{gathered} (y_t(x,t) - \epsilon^2 \triangle y(x,t) - g(x)v(t))(y(x,t) - \psi(x)) = 0 a.e. in Q \\ y_t(x,t) - \epsilon^2 \triangle y(x,t) - g(x)v(t) \geq 0, \\ y(x,t) \geq \psi(x) \ a.e. \ in \ Q, \\ y(x,0) = y_0(x), \ a.e. \ in \ \Omega, \ y(x,t) = 0, \ a.e. \ in \ \Sigma; \end{gathered} \tag{2}$$

here $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$, $\Omega \in R^n$–has compact closure and smooth (from C^∞) (n$-$1)-dimensional boundary $\partial\Omega$, $z(x) \in L_2(\Omega)$, $g(x) \in L_q(\Omega)$, $y_0(x) \in W_0^{2-2/q,q}(\Omega)$, $\psi(x) \in H^2(\Omega)$, $\psi(x) \leq 0$ a.e. on $\partial\Omega$, $y_0 \geq \psi(x)$ a.e. in Ω, $q >$max(n,2), $0 < \epsilon \ll 1$, $\nu =$const> 0, $\triangle$ is the Laplace operator.

The problem (1)-(2) has at least one solution u. Let (y,u) be an pair from the problem (1)-(2). Then ([1]) there exists a function $p \in L_2(0,T;H^1(\Omega)) \bigcap BV([0,T];Y^*)$, $Y = H^s(\Omega) \bigcap H^1(\Omega)$, $s > n/2$ which satisfies the following equations:

$$\begin{gathered} -p_t - \epsilon^2 \triangle p = y(x,t) - z(x) a.e. in \{(x,t) : \ y(x,t) > \psi(x)\}, \\ p(x,t) = 0, \ a.e. \ in \Sigma; \\ p(x,t)(g(x)u(t) + \epsilon^2 \triangle y) = 0 a.e. in \{y = \psi\}, \\ p(x,T) = 0 a.e. in \Omega, \end{gathered} \tag{3}$$

$$u(t) = \begin{cases} -\xi, & (g,p(\cdot,t)) - \nu\xi > 0, \\ & -\nu^{-1}(g,p(\cdot,t)), \\ \xi, & (g,p(\cdot,t)) + \nu\xi < 0, \end{cases} \tag{4}$$

where $(g,p(\cdot,t)) = \int_\Omega g(x)p(x,t)dx$.

2. *FORMAL ASYMPTOTICS.*

We shall find the outer [6] decomposition of the solution of (2)-(4) in the form

$$\bar{y}(x,t) = \sum_{i=0}^{\infty} \epsilon^i \bar{y}_i(x,t);\ \bar{p}(x,t) = \sum_{i=0}^{\infty} \epsilon^i \bar{p}_i(x,t). \tag{5}$$

ASSUMPTION 1. Suppose $z(x),\ y_0(x),\ \psi(x),\ 0 \le g(x) \in C^\infty(\Omega)\bullet$ Zero components of the decomposition (5) are defined like the solution of the problem

$$(\bar{y}_{0_t}(x,t) - g(x)\bar{u}_0(t))(\bar{y}_0(x,t) - \psi(x)) = 0\ in\ Q,$$

$$\bar{y}_{0_t}(x,t) - g(x)\bar{u}_0(t) \ge 0,\ \bar{y}_0(x,t) \ge \psi(x)\ in\ Q,$$
$$\bar{y}_0(x,0) = y_0(x)\ in\ \Omega; \tag{6}$$
$$-\bar{p}_{0_t} = \bar{y}_0(x,t) - z(x)\ in\ \{(x,t) : \bar{y}_0(x,t) > \psi(x)\},$$
$$\bar{p}_0(x,t)g(x)\bar{u}_0(t) = 0\ a.e.\ in\ \{\bar{y}_0 = \psi\},$$
$$\bar{p}_0(x,T) = 0\ in\ \Omega,$$
$$\bar{u}_0(t) = \begin{cases} -\xi, & (g,\bar{p}_0(\cdot,t)) - \nu\xi > 0, \\ & -\nu^{-1}(g,\bar{p}_0(\cdot,t)), \\ \xi, & (g,\bar{p}_0(\cdot,t)) + \nu\xi < 0. \end{cases} \tag{7}$$

Introduce sets

$$Q_0 = \{(x,t) : y(x,t) = \psi(x) a.e.\}, Q_+ = \{(x,t) : y(x,t) > \psi(x) a.e.\}, \tag{8}$$

$$Q = Q_0 \bigcup Q_+,\ Q_0 \bigcap Q_+ = \emptyset,$$

where $y(x,t)$ is the solution of (2)-(4).

Then $\bar{Q}_0,\ \bar{Q}_+$ are their zeroth-order approximations.

ASSUMPTION 2. Suppose that $p(x,t) = 0$ in Q_0 and let Q_0 be a cylinder in R^{n+1} and its base be two-connected domain $(\Omega \backslash \Omega_0)$. Suppose that the outer boundary is equivalent to $\partial\Omega$ and the inner boundary $(\partial\Omega)$ possesses properties of the outer boundary $\bullet$

Then the correlations are fulfilled in $\bar{Q}_+$

$$(x \in \Omega_0,\ t \in T_1^0) : \begin{cases} \dot{\bar{y}}_0 = -g(x)\xi, \\ -\dot{\bar{p}}_0 = \bar{y}_0 - z,\ (g,\bar{p}_0)_0 - \nu\xi > 0; \end{cases} \tag{9}$$

$$(x \in \Omega_0,\ t \in T_2^0) : \begin{cases} \dot{\bar{y}}_0 = g(x)\xi, \\ -\dot{\bar{p}}_0 = \bar{y}_0 - z,\ (g,\bar{p}_0)_0 + \nu\xi < 0; \end{cases} \tag{10}$$

$$(x \in \Omega_0,\ t \in T_3^0) : \begin{cases} \dot{\bar{y}}_0 = -\nu^{-1}g(x)(g,\bar{p}_0)_0, \\ -\dot{\bar{p}}_0 = \bar{y}_0 - z, \end{cases} \tag{11}$$

$$T=\bigcup_{i=1}^{3}T_i^0,\ T_i^0\bigcap T_j^0=\emptyset,\ i\neq j,$$

where $(\cdot,\cdot)$ denotes the scalar products by $\bar{\Omega}_0$.

Let's go over to the problems for $(g,\tilde{y}_0)_0$, $(g,\bar{p}_0)_0$ for the definition of zero components of the control switching moments

$$t\in T_1^0:\begin{cases}(g,\dot{\tilde{y}}_0)_0=-\parallel g\parallel_0^2\xi,\\-(g,\dot{\bar{p}}_0)_0=(g,\tilde{y}_0)_0,\ (g,\bar{p}_0)_0-\nu\xi>0;\end{cases}\tag{12}$$

$$t\in T_2^0:\begin{cases}(g,\dot{\tilde{y}}_0)_0=\parallel g\parallel_0^2\xi,\\-(g,\dot{\bar{p}}_0)_0=(g,\tilde{y}_0)_0,\ (g,\bar{p}_0)_0+\nu\xi<0;\end{cases}\tag{13}$$

$$t\in T_3^0:\begin{cases}(g,\dot{\tilde{y}}_0)_0=-\nu^{-1}\parallel g\parallel_0^2(g,\bar{p}_0)_0,\\-(g,\dot{\bar{p}}_0)_0=(g,\tilde{y}_0)_0,\ \tilde{y}_0=\bar{y}_0-z.\end{cases}\tag{14}$$

The problems (12)-(14) may be solved by dint of phase picture [3] which allows to define the control structure. Then two cases are possible: 1) phase point $((g,\tilde{y}_0)_0,(g,\bar{p}_0))$ doesn't go on bounds, i.e. it belongs to set $P_1=\{0<(g,\bar{p}_0)_0\leq\nu\xi,\ \nu^{-1/2}\parallel g\parallel_0(g,\bar{p}_0)_0<(g,\tilde{y}_0)_0<\infty\}\bigcup\{-\nu\xi\leq(g,\bar{p}_0)_0<0,\ -\infty<(g,\tilde{y}_0)_0<\nu^{-1/2}\times\times\parallel g\parallel_0(g,\bar{p}_0)_0\}$; 2) phase point goes on bounds, i.e. it belongs to set $P_2=\{(g,\bar{p}_0)_0\geq\nu\xi,\ \nu^{-1/2}\parallel g\parallel_0(g,\bar{p}_0)_0\leq(g,\tilde{y}_0)_0<\infty\}\bigcup\{-\nu\xi>(g,\bar{p}_0)_0,\ -\infty<(g,\tilde{y}_0)_0\leq\nu^{-1/2}\parallel g\parallel_0(g,\bar{p}_0)_0\}$. In the case 1) the solution has an appearance:

$$\begin{cases}(g,\tilde{y}_0)_0=(y_0-z,g)_0ch((\nu^{-1/2}\parallel g\parallel_0\times\\\times(T-t))(ch(\nu^{-1/2}\parallel g\parallel_0 T))^{-1},\\(g,\bar{p}_0)_0=\nu^{1/2}\parallel g\parallel_0^{-1}(y_0-z,g)_0\times\\\times sh(\nu^{-1/2}\parallel g\parallel_0(T-t))\times\times(ch(\nu^{-1/2}\parallel g\parallel_0 T))^{-1}\end{cases}\tag{15}$$

on condition that initial data satisfy the inclusion

$$\{(y_0-z,g)_0,\nu^{1/2}\parallel g\parallel_0^{-1}(y_0-z,g)_0th(\nu^{-1/2}\parallel g\parallel_0 T)\}\in P_1.\tag{16}$$

In the case 2) systems (12)-(13) have the solution in $0\leq t\leq\tau_0$ (τ_0 is the moments of descent of control from limitation)

$$\begin{cases}(g,\tilde{y}_0)=\mp\xi\parallel g\parallel_0^2 t+(g,y_0-z)_0,\\(g,\bar{p}_0)=\pm\xi(\nu-1/2\parallel g\parallel_0^2[\tau_0^2-t^2])+(g,y_0-z_0)_0(\tau_0-t).\end{cases}\tag{17}$$

On the segment $t\in(\tau_0,T]$ system (14) has the solution

$$\begin{cases}(g,\tilde{y}_0)_0=\pm\xi\nu^{1/2}\parallel g\parallel_0 ch(\nu^{-1/2}\parallel g\parallel_0(T-t))\times\\\times(sh(\nu^{-1/2}\parallel g\parallel_0(T-\tau_0)))^{-1},\\(g,\bar{p}_0)_0=\pm\xi\nu sh(\nu^{-1/2}\parallel g\parallel_0\times\\\times(T-t))(sh(\nu^{-1/2}\parallel g\parallel_0(T-\tau_0)))^{-1}.\end{cases}\tag{18}$$

Let's regard futher for definition that the condition

$$\begin{gathered}\{(y_0 - z, g)_0, \xi(\nu - 1/2 \parallel g \parallel_0^2 \tau_0^2) + (g, y_0 - z)_0 \tau_0\} \in P_2 \bigcap \\ \{(g, \bar{p}_0)_0 > \nu\xi, \nu^{-1/2} \parallel g \parallel_0 (g, \bar{p}_0)_0 \leq (g, \tilde{y}_0)_0 < \infty\},\end{gathered} \tag{19}$$

which guarantees uniqueness of the solution of equation

$$-\xi \parallel g \parallel_0 (\nu^{1/2} cth(\nu^{-1/2} \parallel g \parallel_0 (T - \tau_0)) + \parallel g \parallel_0 \tau_0) = (g, y_0 - z)_0, \tag{20}$$

is fulfilled. Thus in case 1) the couple $(\bar{y}_0(x,t), \bar{p}_0(x,t))$ is fined from (11). In particular,

$$\begin{gathered}\bar{y}_0(x,t) = y_0(x) - g(x) \parallel g \parallel_0^{-2} (y_0 - z, g)_0 (ch(\nu^{-1/2} \parallel g \parallel_0 T) - \\ -ch(\nu^{-1/2} \parallel g \parallel_0 (T - t)))(ch(\nu^{-1/2} \parallel g \parallel_0 T))^{-1}.\end{gathered} \tag{21}$$

The question about choice of domain $\bar{\Omega}_0$ is solved this way. Let $\tilde{\Omega}_0$ be a set from Ω, which satisfies the condition $y_0(x) > \psi(x)$ and suppose that for any $x \in \tilde{\Omega}_0$ the inequality

$$\bar{y}_0(x,T) > \psi(x) \tag{22}$$

is fulfilled, at that the function $ch(\nu^{-1/2} \parallel g \parallel_0 T) - ch(\nu^{-1/2} \parallel g \parallel_0 (T - t))$ increases monotonically. Systems (9)-(11) are the conditions of optimality in the optimal control problem: find $\bar{u}_0(t) \in U$ such that

$$I_0(v) = \frac{1}{2} \int_0^T \Big(\int_{\bar{\Omega}_0} (\bar{y}_0(x,t) - z(x))^2 dx +$$

$$+ \nu v^2(t)) dt \to min$$

by bounds

$$\dot{\bar{y}}_0(x,t) = g(x) v(t), \bar{y}_0(x,0) = y_0(x).$$

Let $\tilde{\tilde{\Omega}}_0$ be a system of expanded sets which belong to Ω and contain $\tilde{\Omega}_0$ (boundaries of the indicated sets have the properties of the boundary $\partial\Omega$). Then $\bar{\Omega}_0$ is the solution of the optimization problem

$$\begin{gathered}\frac{1}{2} \int_0^T \Big(\int_{\tilde{\tilde{\Omega}}_0} (\bar{y}_0(x,t) - z(x))^2 dx + \int_{\Omega / \tilde{\tilde{\Omega}}_0} (\psi(x) - z(x))^2 dx + \\ + \nu \Big(\int_{\tilde{\tilde{\Omega}}_0} g(x) \bar{p}_0(x,t) \Big)^2 \Big) dt \to min\end{gathered} \tag{23}$$

by bound (16),(22), $\bar{y}_0(x,t)$ is given by the representation (21) and scalar products $(\cdot,\cdot)_0$ are calculated by $\tilde{\tilde{\Omega}}_0$ in all terms.

In case 2) the solution of (9),(11), continuous for $t \in [0,T]$ and smooth for $x \in \bar{\Omega}_0$, is given by the couple $(\bar{y}_0(x,t), \bar{p}_0(x,t))$. In particular, for $t \in [0,\tau_0]$

$$\bar{y}_0(x,t) = -\xi g(x)t + y_0(x),$$

for $t \in (\tau_0, T]$

$$\begin{gathered}\bar{y}_0(x,t) = -\xi g(x)\tau_0 + y_0(x) + \xi\nu^{1/2} \parallel g \parallel_0^{-1} \times \\ \times g(x)(sh(\nu^{-1/2} \parallel g \parallel_0 (T-\tau_0)))^{-1}(ch(\nu^{-1/2} \times \\ \times \parallel g \parallel_0 (T-t)) - ch(\nu^{-1/2} \parallel g \parallel_0 (T-\tau_0))).\end{gathered} \tag{24}$$

The question about choice of domain $\bar{\Omega}_0$ is solved by analogy with preceding case with next changes: the function (23) is minimized by bounds (19),(20),(22) and $\bar{y}_0(x,t)$ is given by representation (24). Let's supplement the solutions $(\bar{y}_0(x,t), \bar{p}_0(x,t))$ on $\partial\bar{\Omega}_0$ by following boundary layer functions $\tilde{y}_0(\bar{t},s,t), \tilde{p}_0(\bar{t},s,t)$ [5,7].

Thereby the solution of (6)-(7) is constructed completely, i.e. zeroth components of decomposition (5) are fined.

ASSUMPTION 3. Suppose the problem's data such that the moment of the control switching $\tau_0 \in (0,T)$ exists and $\partial\bar{\Omega}_0 = \partial\Omega$ •

Let τ be a moment of the control descent from the bound of the initial problem. Let's to find it in the form of an asymptotic series

$$\tau = \sum_{j=0}^{\infty} \epsilon^j \tau_j.$$

The algorithm of the specification of the control switching moment is constructed in [4].

THEOREM. Let's suppose that the assumptions 1-3 are true and (19) takes place. Then the next inequalities hold

$$||grad(y - y^{(N)})||_{L_2(Q)} + ||grad(p - p^{(N)})||_{L_2(Q)} \leq C\epsilon^N,$$

$$||y - y^{(N)}||_{L_2(Q)} + ||p - p^{(N)}||_{L_2(Q)} \leq C\epsilon^{N+1},$$

$$||u - u^{(N)}||_{L_2(0,T)} \leq C\epsilon^{N+1}, \ |I(u) - I(u^{(N)})| \leq C\epsilon^{2(N+1)},$$

where

$$\tau^N = \sum_{i=0}^{N} \epsilon^i \tau_i, \ y^{(N)}(x,t) = \sum_{j=0}^{N} (\bar{y}_j(x,t) + \tilde{y}_j(\bar{t},s,t))\epsilon^j,$$

$$p^{(N)}(x,t) = \sum_{j=0}^{N} (\bar{p}_j(x,t) + \tilde{p}_j(\bar{t},s,t))\epsilon^j,$$

$$u^{(N)}(t) = \begin{cases} -\xi, & 0 \leq t \leq \tau^N, \\ -\nu^{-1}(g, p^{(N)}(.,t)), & \tau^N \leq t \leq T. \end{cases}$$

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