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C^∞-REGULARITY OF NON-LIPSCHITZ HEAT SEMIGROUPS ON NONCOMPACT RIEMANNIAN MANIFOLDS

We obtain the applications of approach [2, 5, 6] to the high order regularity of solutions to the parabolic Cauchy problem with globally non-Lipschitz coefficients growing at the infinity of a noncompact manifold.

In comparison to [2], where the semigroup properties were studied by application of nonlinear estimates on variations with use of local arguments of [11], i.e. for manifolds with the C^2 metric distance function, the developed below approach works for the general noncompact manifold with possible non-unique geodesics between distant points.

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1.Introduction.

Below we discuss C^∞-regularity with respect to the space parameter x for solutions of the second order heat parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \left\{ \tfrac{1}{2}\sum_{\sigma=1}^{d} \left(\langle A_\sigma(x), \frac{\partial}{\partial x}\rangle \right)^2 + \langle A_0(x), \frac{\partial}{\partial x}\rangle \right\} u(t,x), \qquad (t,x)\in I\!R_+ \times M \tag{1}$$

with initial data $u(0,x) = f(x)$. The nonlinear, growing on the infinity coefficients $A_\sigma,\ A_0$ represent C^∞ smooth vector fields on the noncompact oriented C^∞-smooth complete connected Riemannian manifold without boundary M.

To obtain C^∞-regularity properties of solutions to heat equations (1) in the spaces of continuously differentiable functions we use that the solution to (1) can be represented

$$u(t,x) = (P_t f)(x) = \mathbf{E} f(y_t^x) \tag{2}$$

as a mean $\mathbf{E}$ of solution to the heat diffusion equation in the Ito-

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Stratonovich form

$$\delta y_t^x = A_0(y_t^x)dt + \sum_{\sigma=1}^{d} A_\sigma(y_t^x)\delta W_t^\sigma, \quad y_0^x = x. \tag{3}$$

Here W^σ denote the independent Wiener processes on $\mathbb{R}^d$, the mean $\mathbf{E}$ is taken with respect to the corresponding Wiener measure on space $\Omega = C_0([0,\infty), \mathbb{R}^d)$. Under solutions to (3) it is understood a continuous adapted integrable random process $\mathbb{R}_+ \times M \ni (t,x) \to y_t^x \in M$ such that $\forall f \in C_0^\infty(M)$

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x)ds + \sum_{\sigma=1}^{d} \int_0^t (A_\sigma f)(y_s^x)\delta W_s^\sigma. \tag{4}$$

Since $f(y_t^x)$ and $(A.f)(y_t^x)$ are $\mathbb{R}^1$-valued process, (4) represents Stratonovich equation on real line $\mathbb{R}^1$.

The approach of diffusion processes relates the C^∞ properties of semigroup P_t with the regularity properties of process y_t^x since

1. from representation (2) and majorant theorem it is evident that for continuous with respect to the initial data process $x \to y_t^x$ from $f \in C_b(M)$ follows that the solution $u(t,x) = (P_t f)(x)$ to (1) is also a continuous bounded function $u(t,x) \in C_b(M)$ for all $t > 0$

and

2. due to (2) the derivatives of solution $u(t,x)$ with respect to the space parameter $x \in M$ can be directly expressed via the derivatives of initial function f and derivatives of process y_t^x with respect to the initial data x.

The already known approaches to the high order regularity of solutions to (1), e.g. [8] and references within, were based on the interpretation of the high order derivatives of process $x \to y_t^x$ with respect to the initial data x as elements $y_t^{(n)}(x) = T^{(n)} y_t^x$ of the high order tangent bundles $T^n M$. However, due to the complicate structure of the high order tangent bundles, the actual study of the high order regularity of process y_t^x and its semigroup P_t was conducted in the local coordinate vicinities of manifold, leading to the globally Lipschitz assumptions on the coefficients A_σ, A_0 and boundedness of curvature and all their derivatives. In comparison to the case of linear manifold $M = I\!R^n$, when under some monotone assumptions on the coefficients of diffusion equation [10, 12] it is possible to have a nonlinear growth of coefficients on the infinity, the elaborated approach to the C^∞-regularity of diffusions on noncompact manifolds has not permitted to single out some kind of monotone conditions.

In [2, 3] we noticed that, since, evolving in time, process y_t^x travels through different coordinate vicinities of manifold, its first order derivative with respect to the initial data $\frac{\partial y_t^x}{\partial x} = \left\{ \frac{\partial (y_t^x)^m}{\partial x^k} \right\}_{m,k=1}^{\dim M}$ represents a covector field with respect to the local coordinates (x^k) in the vicinity of initial point x and becomes a vector field with respect to the local coordinates (y^m) in the vicinity where now travels process y_t^x, i.e. $\frac{\partial y_t^x}{\partial x} \in T^{1,0}_{y_t^x} M \otimes T^{0,1}_x M$.

To preserve this tensorial invariance property with respect to the coordinate systems (y^m) in the image of flow $x \to y_t^x$, the new high order variations $\left\{ \mathbb{V}^{(n)} y_t^x \right\}$ of process y_t^x were introduced

$$\text{1}^{st}\ \text{variation.} \qquad \left\{ \mathbb{V}^{(1)} y_t^x \right\}_k^m = \frac{\partial (y_t^x)^m}{\partial x^k},$$

$$\text{high variations.} \qquad \left\{ \mathbb{V}^{(n+1)} y_t^x \right\}_{k,j_1,\dots,j_n}^m = \tag{5}$$

$$= \nabla_k^x \left\{ \mathbb{V}^{(n)} y_t^x \right\}_{j_1,\dots,j_n}^m + \Gamma_{p\ q}^{\ m}(y_t^x) \left\{ \mathbb{V}^{(n)} y_t^x \right\}_{j_1,\dots,j_n}^p \frac{\partial (y_t^x)^q}{\partial x^k},$$

where $\nabla_k^x \left\{ \mathbb{V}^{(n)} y_t^x \right\}_{j_1,\dots,j_n}^m$ denotes a classical covariant derivative on variable x

$$\nabla_k^x \left\{ \mathbb{V}^{(n)} y_t^x \right\}_\gamma^m = \partial_k^x \left\{ \mathbb{V}^{(n)} y_t^x \right\}_\gamma^m - \sum_{j \in \gamma} \Gamma_{k\ j}^{\ h}(x) \left\{ \mathbb{V}^{(n)} y_t^x \right\}_{\gamma|_{j=h}}^m \tag{6}$$

and $\left\{ \mathbb{V}^{(n)} y_t^x \right\}_{\gamma|_{j=h}}^m$ means substitution of index j in multi-index $\gamma = \{j_1, \dots, j_n\}$ by h. Above indexes m, p, q correspond to the coordinates in the vicinities, where travels process y_t^x, indexes k, j, h to the coordinate vicinities of initial data x.

In comparison to the approach of high order tangent bundles and due to the presence of additional connection term $\Gamma(y_t^x)$ in (5), the n^{th} order variation now represents a vector field with respect to variable y_t^x and n^{th} order covariant field with respect to variable x

$$\mathbb{V}^{(n)} y_t^x = \{\mathbb{V}_\gamma (y_t^x)^m\}_{|\gamma|=n} \in T_{y_t^x} M \otimes (T_x^* M)^{\otimes n}$$

i.e. it is understood as a tensor and does not belong to the high order tangent bundle $T^{(n)}M$.

In this article we are going to apply results of [5, 6] to the high order regularity of semigroup. The main tool for research gives the following relation between high order covariant derivatives of semigroup and initial function [2]

$$\nabla_x^{(n)} P_t f(x) = \sum_{j_1+\ldots+j_\ell=n, \ell\geq 1} \mathbf{E} \, \langle \nabla_{y_t^x}^{(\ell)} f(y_t^x), \mathbb{V}^{(j_1)} y_t^x \otimes \ldots \otimes \mathbb{V}^{(j_\ell)} y_t^x \rangle_{T^{0,\ell}_{y_t^x} M}, \tag{7}$$

where arise variations $\mathbb{V}_x^{(n)} y_t^x$.

Unfortunately, the approach of nonlinear estimates on variations [2] works only for manifolds with smooth Riemannian structures, in particular, when the metric distance function is twice continuously differentiable. It is not so for many C^∞ manifolds with non-unique geodesics between distant points, therefore the approach of [2] should be modified.

The main result is the following. Suppose that the coefficients of diffusion equation (3) and curvature of manifolds fulfill the following conditions:

- **coercitivity**: $\exists\, o \in M$ such that $\forall C \in \mathbb{R}_+ \ \exists\, K_C \in \mathbb{R}^1$ such that $\forall x \in M$

$$\langle \widetilde{A_0}(x), \nabla^x \rho^2(o,x) \rangle + C \sum_{\sigma=1}^d \|A_\sigma(x)\|^2 \leq K_C(1+\rho^2(o,x)), \tag{8}$$

  where $\rho(x,o)$ denotes the shortest geodesic distance between points $x, o \in M$;

- **dissipativity**: $\forall C, C' \in \mathbb{R}_+ \ \exists\, K_C \in \mathbb{R}^1$ such that $\forall x \in M$,

$$\begin{gathered} \forall h \in T_x M \quad \langle \nabla \widetilde{A_0}(x)[h], h\rangle + C \textstyle\sum_{\sigma=1}^d \|\nabla A_\sigma(x)[h]\|^2 - \\ -C' \textstyle\sum_{\sigma=1}^d \langle R_x(A_\sigma(x), h) A_\sigma(x), h\rangle \leq K_C \|h\|^2, \end{gathered} \tag{9}$$

  where $\widetilde{A_0} = A_0 + \frac{1}{2} \sum\limits_{\sigma=1}^d \nabla_{A_\sigma} A_\sigma$ and

$$[R(A,B)C]^m = R_{i\ jk}^{\ m} A^i B^j C^k$$

denotes curvature operator, related with curvature tensor

$$R_{i\ jk}^{\ m} = \frac{\partial \Gamma_{i\ j}^{\ m}}{\partial x^k} - \frac{\partial \Gamma_{i\ k}^{\ m}}{\partial x^j} + \Gamma_{i\ j}^{\ \ell}\Gamma_{\ell\ k}^{\ m} - \Gamma_{i\ k}^{\ \ell}\Gamma_{\ell\ j}^{\ m}.$$

Due to the presence of additional curvature term conditions (8)-(9) generalize the classical dissipativity and coercitivity conditions [10, 12] from the linear Euclidean space to manifold and represent a kind of monotonicity condition for diffusion processes.

- **nonlinear behaviour of coefficients and curvature**: for any n there are constants $\mathbf{k}_\bullet$ such that for all $j = 1,..,n$ and $\forall\, x \in M$

$$\begin{aligned} \|(\nabla)^j \widetilde{A_0}(x)\| &\le (1+\rho(x,o))^{\mathbf{k}_0}, \\ \|(\nabla)^j A_\alpha(x)\| &\le (1+\rho(x,o))^{\mathbf{k}_\alpha}, \\ \|(\nabla)^j R(x)\| &\le (1+\rho(x,o))^{\mathbf{k}_R}. \end{aligned} \tag{10}$$

Let $\vec{q}_{\mathbf{k}} = (q_0, q_1, ..., q_n),\ q_i \ge 1$ be a family of monotone functions on $I\!R_+$ of polynomial behaviour, that fulfill hierarchy

$$\forall\, i \ge 1 \quad q_i(b)(1+b)^{\mathbf{k}/2} \le q_{i+1}(b) \quad \forall\, b \ge 0, \tag{11}$$

related with some parameter $\mathbf{k}$.

Denote by $C^n_{\vec{q}(\mathbf{k})}(M)$ the space of n-times continuously covariantly differentiable functions on M, equipped with a norm

$$\|f\|_{C^n_{\vec{q}(\mathbf{k})}(M)} = \max_{i=0,\dots,n} \sup_{x\in M} \frac{\|(\nabla^x)^i f(x)\|}{q_i(\rho^2(x,o))}. \tag{12}$$

Theorem 1. *Under conditions (8)-(10) for any $n \in I\!N$ there is $\mathbf{k}$ such that the scale of spaces $C^n_{\vec{q}(\mathbf{k})}(M)$ is preserved under the action of semigroup*

$$\forall\, t \ge 0 \quad P_t : C^n_{\vec{q}(\mathbf{k})}(M) \to C^n_{\vec{q}(\mathbf{k})}(M)$$

and there are constants K, M such that

$$\forall\, f \in C^n_{\vec{q}}(M) \qquad \|P_t f\|_{C^n_{\vec{q}}(M)} \le K e^{Mt} \|f\|_{C^n_{\vec{q}}(M)}. \tag{13}$$

Proof is conducted in the following Sections.

2. Preliminary study of C^∞-regularity of process y_t^x with respect to the initial data x.

Since by (7) the existence of the high order derivatives of semigroup P_t is related with C^∞-regularity of process y_t^x with respect to x, we first discuss the necessary regular properties of variations.

Let us introduce a necessary definition for the parallel transport $\overset{h,y_t^h}{\mathbb{T}}{}_a^b$ of high order variations. It is specially designed in order to preserve the tensorial transformation law of $T^{1,0}_{y_t^{h(z)}}M\otimes T^{0,n}_{h(z)}M$-tensors in the both domain and image of diffusion flow $x\to y_t^x$, when such tensors move along random path $[a,b]\ni z\to (h(z),y_t^{h(z)})\in M\times M$.

Definition 2. The parallel transport of tensor $u_{(h(a),y_t^{h(a)})}\in T^{p,q}_{h(a)}M\otimes T^{r,s}_{y_t^{h(a)}}M$ from point $(h(a),y_t^{h(a)})$ along path $(h(\cdot),y_t^{h(\cdot)})\in Lip([a,b],M)$ represents a $T^{p,q}_{h(z)}M\otimes T^{r,s}_{y_t^{h(z)}}M$-tensor at each point $(h(z),y_t^{h(z)}),\ z\in[a,b]$ of this path. It is denoted by $\overset{h,y_t^h}{\mathbb{T}}{}^{(h(z),y_t^{h(z)})}_{(h(a),y_t^{h(a)})}u_{(h(a),y_t^{h(a)})}=\Psi(z)$ and for its absolute derivative

$$\frac{\mathbb{D}}{\mathbb{D}z}\Psi^{(i/\alpha)}_{(j/\beta)}(z)\overset{def}{=}\frac{\partial}{\partial z}\Psi^{(i/\alpha)}_{(j/\beta)}(z)+$$

$$+\sum_{s=1}^{i}\Gamma_{k\ \ell}^{\ i_s}(h(z))\,\Psi^{i_1,..,i_{s-1},k,i_{s+1},..,i_p/\alpha}_{(j/\beta)}(z)[h'(z)]^\ell-$$

$$-\sum_{s=1}^{j}\Gamma_{j_s\ \ell}^{\ k}(h(z))\,\Psi^{(i/\alpha)}_{j_1,..,j_{s-1},k,j_{s+1},..,j_q/\beta}(z)+$$

$$+\sum_{\ell=1}^{r}\Gamma_{m\ n}^{\ \alpha_\ell}(y_t^{h(z)})\Psi^{(i/\alpha_1,\dots,\alpha_{\ell-1},m,\alpha_{\ell+1},\dots,\alpha_r)}_{(j/\beta)}(z)\frac{\partial(y_t^{h(z)})^n}{\partial h(z)^k}[h'(z)]^k-$$

$$-\sum_{\ell=1}^{s}\Gamma_{\beta_\ell\ n}^{\ m}(y_t^{h(z)})\Psi^{(i/\alpha)}_{(j/\beta_1,\dots,\beta_{\ell-1},m,\beta_{\ell+1},\dots,\beta_s)}(z)\frac{\partial(y_t^{h(z)})^n}{\partial h(z)^k}[h'(z)]^k$$

the norm $||\frac{\mathbb{D}}{\mathbb{D}z}\Psi(z)||_{T^{p,q}_{h(z)}M\otimes T^{r,s}_{y_t^{h(z)}}M}=0$ vanishes in $L^\infty([a,b])$ for a.e. random $\omega\in\Omega$. Here multi-indexes $(i)=(i_1,...,i_p)$, $(j)=(j_1,...,j_q)$ correspond to the $T^{p,q}_{h(z)}$-tensorness of $\Psi(z)$, correspondingly $(\alpha)=(\alpha_1,...,\alpha_r)$, $(\beta)=(\beta_1,...,\beta_s)$ to $T^{r,s}_{y_t^{h(z)}}$-tensorness of $\Psi(z)$ in the domain and image of mapping $x\to y_t^x$.

Remark that the first two lines in the definition of the absolute derivative $\frac{I\!\!D}{I\!\!Dz}\Psi^{(i/\alpha)}_{(j/\beta)}(z)$ along path $\{h(z), y_t^{h(z)}\}_{z\in[a,b]}$ correspond to the classical absolute derivative $\frac{D}{Dz}\Psi^{(i/\alpha)}_{(j/\beta)}(z)$ along path $\{h(z)\}_{z\in[a,b]}$. The remaining two lines make the resulting expression to become the invariantly defined tensor with respect to the coordinate transformations in vicinities, where travels process $\{y_t^{h(z)}\}_{z\in[a,b]}$.

Using the autoparallel property of the Riemannian connection

$$\partial_k g_{ij}(x) = \Gamma_k{}^\ell{}_i(x) g_{\ell j}(x) + \Gamma_k{}^\ell{}_j(x) g_{i\ell}(x), \tag{14}$$

it is easy to check that the derivative of scalar product of $T_x^{p,q}M \otimes T^{r,s}_{y_t^x}M$-tensors can be expressed in terms of the new type absolute derivatives

$$\begin{aligned}\frac{d}{dz}\langle u(h(z)), v(h(z))\rangle_{T^{p,q}_{h(z)}M\otimes T^{r,s}_{y_t^{h(z)}}M} =\\ = \langle \frac{I\!\!D}{I\!\!Dz} u(h(z)), v(h(z))\rangle_{T^{p,q}_{h(z)}M\otimes T^{r,s}_{y_t^{h(z)}}M} +\\ + \langle u(h(z)), \frac{I\!\!D}{I\!\!Dz} v(h(z))\rangle_{T^{p,q}_{h(z)}M\otimes T^{r,s}_{y_t^{h(z)}}M}\end{aligned} \tag{15}$$

and parallel transport operator $\overset{h,y_t^h}{I\!\!I}$ constitutes a group $\forall\, c, d, e \in [a,b]$ $\overset{h,y_t^h}{I\!\!I}{}^e_d \overset{h,y_t^h}{I\!\!I}{}^d_c = \overset{h,y_t^h}{I\!\!I}{}^e_c$.

Then, taking any mixed tensor $\psi_{h(a)} \in T^{p,q}_{h(a)} \otimes T^{r,s}_{y_t^{h(a)}}$ at point $(h(a), y_t^{h(a)})$ we have

$$\frac{d}{dz}\langle \psi_{h(a)}, \overset{h,y_t^h}{I\!\!I}{}^a_z u_{h(z)}\rangle_{T^{p,q}_{h(a)}\otimes T^{r,s}_{y_t^{h(a)}}} = \frac{d}{dz}\langle \overset{h,y_t^h}{I\!\!I}{}^z_a \psi_{h(a)}, u_{h(z)}\rangle_{T^{p,q}_{h(z)}\otimes T^{r,s}_{y_t^{h(a)}}} =$$

$$= \langle \overset{h,y_t^h}{I\!\!I}{}^z_a \psi_{h(a)}, \frac{I\!\!D}{I\!\!Dz} u_{h(z)}\rangle_{T^{p,q}_{h(z)}\otimes T^{r,s}_{y_t^{h(a)}}} = \langle \psi_{h(a)}, \overset{h,y_t^h}{I\!\!I}{}^a_z \left[\frac{I\!\!D}{I\!\!Dz} u_{h(z)}\right]\rangle_{T^{p,q}_{h(a)}\otimes T^{r,s}_{y_t^{h(a)}}},$$

where we used that the derivative of parallel transport vanishes $\frac{I\!\!D}{I\!\!Dz}\overset{h,y_t^h}{I\!\!I}{}^z_a \psi_{h(a)} = 0$.

Integrating on variable $z \in [a,b]$ we obtain

$$< \psi_{h(a)}, \int_a^b \overset{h,y_t^h}{I\!\!I}{}^a_z \left[\frac{I\!\!D}{I\!\!Dz} u_{h(z)}\right] dz >= \int_a^b \frac{d}{dz} < \psi_{h(a)}, \overset{h,y_t^h}{I\!\!I}{}^a_z u_{h(z)} > dz =$$

$$=< \psi_{h(a)}, \overset{h,y_t^h}{\mathbf{T}}{}^a_z u_{h(z)} > \Big|_{z=a}^{z=b} =< \psi_{h(a)}, \overset{h,y_t^h}{\mathbf{T}}{}^a_b u_{h(b)} - u_{h(a)} > .$$

Since $\psi_{h(a)}$ was arbitrary, this implies the invariant formula for the increment of mixed tensors along Lipschitz paths

$$\overset{h,y_t^h}{\mathbf{T}}{}^a_b u_{h(b)} - u_{h(a)} = \int_a^b \overset{h,y_t^h}{\mathbf{T}}{}^a_z \left[\frac{\mathbb{D} u_{h(z)}}{\mathbb{D} z} \right] dz \tag{16}$$

and, in particular, recovers a sense of the new type mixed absolute derivative of $T_h^{p,q} \otimes T_{y_t^h}^{r,s}$-tensors

$$\frac{d}{dz} \overset{h,y_t^h}{\mathbf{T}}{}^a_z u_{h(z)} = \overset{h,y_t^h}{\mathbf{T}}{}^a_z \left[\frac{\mathbb{D} u_{h(z)}}{\mathbb{D} z} \right] \quad \text{or}$$
$$\frac{\mathbb{D} u_{h(z)}}{\mathbb{D} z} = \overset{h,y_t^h}{\mathbf{T}}{}^z_a \left[\frac{d}{dz} \overset{h,y_t^h}{\mathbf{T}}{}^a_z u_{h(z)} \right], \quad z \in [a,b]. \tag{17}$$

Therefore, since the high order variations $\mathbb{V}^{(n)} y_t^x$ represent a particular case of $T_x^{0,n} \otimes T_{y_t^x}^{1,0}$-tensors, they should be related by similar to (16) formulas. To find the sufficient monotone conditions on the existence of the high order derivatives $\mathbb{V}^{(n)} y_t^x$ of process $x \to y_t^x$ we first construct the solutions $y_{t,x}^{(n)}$ of the associated with (3) variational system and then verify that they represent the high order $\mathbb{V}$-derivatives: $y_{t,x}^{(n)} = \mathbb{V}^{(n)} y_t^x, \ \forall n \in \mathbb{N}$.

The main result about the C^∞-regularity of process y_t^x follows. Here we also precise the influence of nonlinearity parameter $\mathbf{k}$ (10) on the growth of high order derivatives.

Lemma 3. *Under the conditions of Theorem 1 the new type variations are related by a.e. integral formulas* $\forall f \in C_0^\infty(M), \forall n \in \mathbb{N}$

$$f(y_t^{h(b)}) - f(y_t^{h(a)}) = \int_a^b < \nabla f(y_t^{h(z)}), \mathbb{V} y_t^{h(z)}[h'(z)] >_{T_{y_t^{h(z)}}} dz, \tag{18}$$

$$\mathbb{V}^{(n)} y_t^{h(b)} - \overset{h,y_t^h}{\mathbf{T}}{}^b_a \left[\mathbb{V}^{(n)} y_t^{h(a)} \right] = \int_a^b \overset{h,y_t^h}{\mathbf{T}}{}^b_z \left[\left[\mathbb{V}^{(n+1)} y_t^{h(z)} \right] [h'(z)] \right] dz \tag{19}$$

for any Lipschitz continuous path $h \in Lip([a,b], M)$.

Moreover, they fulfill estimates

$$\forall n \in \mathbb{N} \ \exists M_n \quad \mathbf{E} || \mathbb{V}^{(n)} y_t^x ||^{2q} \le e^{2qM_n t} (1 + \rho^2(x,o))^{q(n-1)\mathbf{k}}. \tag{20}$$

Remark that estimate (20) actually replaces the tool of nonlinear estimate on variations, discussed e.g. in [2], for manifolds with not everywhere C^2-smooth square of metric distance function $\rho^2(x,z)$, $(x,z)\in M\times M$.

Proof. First note that under conditions of Theorem 1 there is a unique strong solution y_t^x to equation (3), which fulfills estimates on the boundedness and continuity: $\exists\, M\ \forall\, q\ge 1$

$$\begin{array}{ll} [5,\ \text{Th.5}]: & \mathbf{E}\left(1+\rho^2(y_t^x,o)\right)^q \le e^{Mqt}\left(1+\rho^2(x,o)\right)^q, \\ [6,\ \text{Th.6}]: & \mathbf{E}\rho^{2q}(y_t^x,y_t^z)\le e^{Mqt}\rho^{2q}(x,z). \end{array} \tag{21}$$

Moreover, relation (18) was proved in [6, Th.8].

It remains to demonstrate (19) and estimate (20). Remark that estimate (20) for $i=1$ gives an alternative proof of [6, Th.7].

Recall that the differential equations on variations have form [2, Th.9]

$$\delta([\mathbb{V}y_t^x]^m_\gamma) = -\Gamma_{p\ q}^{\ m}(y_t^x)\,[\mathbb{V}y_t^x]^p_\gamma\,\delta y^q + M_{\gamma\ \alpha}^{\ m}\delta W^\alpha + N^m_\gamma dt \tag{22}$$

with coefficients $M_{\gamma\ \alpha}^{\ m}$, N^m_γ, determined by

1. recurrence base for $|\gamma|=1,\ \gamma=\{k\}$:

$$M_{k\ \alpha}^{\ m} = \nabla_\ell A^m_\alpha(y_t^x)\mathbb{V}_k y^\ell,\quad N^m_k = \nabla_\ell A^m_0(y_t^x)\mathbb{V}_k y^\ell; \tag{23}$$

2. recurrence step

$$M_{\gamma\cup\{k\}\ \alpha}^{\ m} = \mathbb{V}_k M_{\gamma\ \alpha}^{\ m} + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma y^p)(\mathbb{V}_k y^\ell)A^q_\alpha, \tag{24}$$

$$N^m_{\gamma\cup\{k\}} = \mathbb{V}_k N^m_\gamma + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma y^p)(\mathbb{V}_k y^\ell)A^q_0. \tag{25}$$

The unique strong solution of variational system (22) can be constructed either by gluing together the solutions of variational equations, localized to the local coordinate vicinities of $U\subset M$ on the random time intervals of entering and leaving such vicinities, or with the use of monotone approximations of system (22), similar to [1].

Taking the differential of norm of variational process we have [2, Lemma 10]

$$d\|\mathbb{V}^{(i)}y_t^x\|^2 = g^{\gamma\varepsilon}(x)\,\{\,g_{mn}(\mathbb{V}_\gamma y^m\,M_{\varepsilon\ \alpha}^{\ n} + \mathbb{V}_\varepsilon y^n\,M_{\gamma\ \alpha}^{\ m})dW^\alpha +$$

$$+g_{mn}(\mathbb{V}_\gamma y^m\,N^n_\varepsilon + \mathbb{V}_\varepsilon y^n\,N^m_\gamma + M_{\gamma\ \alpha}^{\ m}M_{\varepsilon\ \alpha}^{\ n})dt +$$

$$+\frac{1}{2} g_{mn}(\mathbb{V}_\gamma y^m \, P^n_\varepsilon + \mathbb{V}_\varepsilon y^n \, P^m_\gamma \,) dt \; \} \tag{26}$$

with $|\gamma| = |\varepsilon| = i$ and expressions P^m_γ are recurrently defined by

$$P^m_k = \nabla^y_\ell \nabla_{A_\alpha} A^m_\alpha \cdot \mathbb{V}_k y^\ell - R(A_\alpha, \mathbb{V}_k y) A_\alpha; \tag{27}$$

$$\begin{gathered} P^m_{\gamma \cup \{k\}} = \mathbb{V}_k P^m_\gamma + 2 R_p{}^m{}_{\ell q} M_\gamma{}^p{}_\alpha (\mathbb{V}_k y^\ell) A^q_\alpha + \\ +(\nabla_s R_p{}^m{}_{\ell q})(\mathbb{V}_\gamma y^p)(\mathbb{V}_k y^\ell) A^q_\alpha A^s_\alpha + R_p{}^m{}_{\ell q}(\mathbb{V}_\gamma y^p)(\mathbb{V}_k A^\ell_\alpha) A^q_\alpha + \\ +R_p{}^m{}_{\ell q}(\mathbb{V}_\gamma y^p)(\mathbb{V}_k y^\ell)(\nabla_{A_\alpha} A_\alpha). \end{gathered} \tag{28}$$

Since in (28) $P^m_{\gamma \cup \{k\}} = \mathbb{V}_k P^m_\gamma + ...$, the high order coefficient permits representation

$$\begin{aligned} P^m_\gamma &= \nabla_\ell \nabla_{A_\alpha} A^m_\alpha \cdot \mathbb{V}_\gamma y^\ell - R(A_\alpha, \mathbb{V}_\gamma y) A_\alpha + \\ &+ \sum_{\beta_1 \cup .. \cup \beta_s = \gamma, \; s \geq 2} K_{\beta_1,..,\beta_s}(\mathbb{V}_{\beta_1} y, ..., \mathbb{V}_{\beta_s} y) \end{aligned}$$

with coefficients $K_{\beta_1,...,\beta_s}$, depending on A_0, A_α, R and their covariant derivatives.

In the same way, due to (23)-(25), we have similar asymptotic

$$M_\gamma{}^m{}_\alpha = \nabla^y_\ell A^m_\alpha [\mathbb{V}_\gamma y^\ell] + \sum_{\beta_1 \cup ... \cup \beta_s = \gamma, \; s \geq 2} K'_{\beta_1,...,\beta_s}(\mathbb{V}_{\beta_1} y, ..., \mathbb{V}_{\beta_s} y); \tag{29}$$

$$N_\gamma{}^m = \nabla^y_\ell A^0_\alpha [\mathbb{V}_\gamma y^\ell] + \sum_{\beta_1 \cup ... \cup \beta_s = \gamma, \; s \geq 2} K''_{\beta_1,...,\beta_s}(\mathbb{V}_{\beta_1} y, ..., \mathbb{V}_{\beta_s} y)$$

with multilinear coefficients K', K'', depending on A_0, A_α, R and their covariant derivatives.

Therefore from (26) the principal part of differential is

$$\begin{gathered} d\|\mathbb{V}^{(i)} y^x_t\|^2 = 2\langle \mathbb{V}^{(i)} y, \nabla^y_\ell A_\alpha [\mathbb{V}^{(i)} y^\ell] \rangle dW^\alpha + \\ +\{2\langle \mathbb{V}^{(i)} y, \nabla^y_\ell \widetilde{A_0} [\mathbb{V}^{(i)} y^\ell] \rangle + \sum_{\alpha=1}^d \|\nabla A_\alpha [\mathbb{V}^{(i)} y] \,\|^2 - \\ - \sum_{\alpha=1}^d \langle R(A_\alpha, \mathbb{V}^{(i)} y) A_\alpha, \mathbb{V}^{(i)} y \rangle \; \} dt + \\ + \sum_{j_1+...+j_s=i, \; s \geq 2} \langle \mathbb{V}^{(i)} y, \; \{ K^1_{j_1,...,j_s,\alpha}(\mathbb{V}^{(j_1)} y, ..., \mathbb{V}^{(j_s)} y) dW^\alpha + \\ +K^2_{j_1,...,j_s}(\mathbb{V}^{(j_1)} y, ..., \mathbb{V}^{(j_s)} y) dt \} \rangle, \end{gathered} \tag{30}$$

i.e. the dissipativity condition arises in the principal part. Like before the coefficients K^1, K^2 depend on covariant derivatives of A_0, A_α, R.

Using asymptotic (30) we come to the dissipativity condition (9) in principal part and additional terms with lower order variations

$$h(t) = \mathbf{E}||\nabla^{(i)} y_t^x||^{2q} \le h(0)+$$

$$+K\mathbf{E}\int\int_0^t \|\nabla^{(i)} y_t^x\|^{2(q-1)}\{\text{dissipativity}\}_{C,C'}(\nabla^{(i)} y_t^x, \nabla^{(i)} y_t^x)dt+$$

$$+\sum_{j_1+...+j_s=i,\ s\ge 2} \mathbf{E}\int_0^t \|\nabla^{(i)} y_t^x\|^{2(q-1)} \langle \nabla^{(i)} y, K_{j_1,...,j_s}(\nabla^{(j_1)} y, ..., \nabla^{(j_s)} y)\rangle dt. \tag{31}$$

By inequality $|x^{q-1}y| \le |x|^q/q + (q-1)|y|^q/q$ and (10)

$$\mathbf{E}\|\nabla^{(i)} y\|^{2(q-1)} \left| K_{i;j_1,...,j_s}(\nabla^{(i)} y; \nabla^{(j_1)} y, ..., \nabla^{(j_s)} y)\right| \le$$

$$\le \mathbf{E}(1+\rho^2(o, y_t^x))^{\mathbf{k}/2} \|\nabla^{(i)} y\|^{2q-1} \|\nabla^{(j_1)} y\| ... \|\nabla^{(j_s)} y\| \le$$

$$\le C\mathbf{E}\|\nabla^{(i)} y\|^{2q} + C'\mathbf{E}(1+\rho^2(o, y_t^x))^{q\mathbf{k}} \|\nabla^{(j_1)} y\|^{2q} ... \|\nabla^{(j_s)} y\|^{2q}$$

with $\mathbf{k}$ determined by nonlinearity parameters (10).

To transform the last term let us use the inductive assumption (20) for lower order variations. By Gronwall-Bellmann and Hölder inequalities (31) implies

$$h(t) \le e^{Ct} h(0) + \sum_{j_1+...+j_s=i,\ s\ge 2} C' \int_0^t e^{C(t-s)} \mathbf{E}(1+\rho^2(o, y_t^x))^{q\mathbf{k}} \times$$

$$\times ||\nabla^{(j_1)} y_t^x||^{2q} \cdot ... \cdot ||\nabla^{(j_s)} y_t^x||^{2q} \le$$

$$\le e^{Ct} h(0) + \sum_{j_1+...+j_s=i,\ s\ge 2} e^{(C+C')t} \sup_{s\in[o,t]} \Big(\mathbf{E}(1+\rho^2(o, y_t^x))^{q\mathbf{k} r_0}\Big)^{1/r_0} \times$$

$$\times \prod_{p=1}^{s} \Big(\mathbf{E}||\nabla^{(j_p)} y_t^x||^{2q r_p}\Big)^{1/r_p} \le$$

$$\le e^{(C+C'+2qM)t} \sum_{j_1+...+j_s=i,\ s\ge 2} (1+\rho^2)^{q\mathbf{k}} \prod_{p=1}^{s} (1+\rho^2)^{q(j_p-1)\mathbf{k}} \le \tag{32}$$

$$\le e^{2qM't}(1+\rho^2(o, y_t^x))^{q(i-1)\mathbf{k}},$$

which leads to (20).

Finally, let us show how to prove (19). Making assumption that the differential equation on the parallel transport $\overset{h,y_t^h}{\mathbb{T}}{}_a^z y_{t,h(a)}^{(n)}$ of the high order variation has similar to (22) form:

$$\delta\left[\overset{h,y_t^h}{\mathbb{T}}{}_a^z\, y_{t,h(a)}^{(n)}\right] = -\Gamma\left(\overset{h,y_t^h}{\mathbb{T}}{}_a^z\, y_{t,h(a)}^{(n)}, \delta y_t^{h(b)}\right) + \sum_\alpha K_\alpha^{(n)}(z)\delta W^\alpha + L^{(n)}(z)dt, \tag{33}$$

the following relations are found: $\forall\, z \in [a,b]$

$$\begin{cases} \dfrac{\mathbb{D}}{\mathbb{D}z} K_\alpha^z = R(\Psi^z, A_\alpha(y_t^{h(z)}))y_{t,h(z)}^{(1)}[h'(z)]; \\ \dfrac{\mathbb{D}}{\mathbb{D}z} L^z = R(\Psi^z, A_0(y_t^{h(z)}))y_{t,h(z)}^{(1)}[h'(z)]. \end{cases} \tag{34}$$

with the initial data $K_\alpha^{(n)}(a) = M_\alpha^{(n)},\ L^{(n)}(a) = N^{(n)}$ defined in (22) due to $\overset{h,y_t^h}{\mathbb{T}}{}_a^a = Id$. These relations are proved in analogue to the proof of [3, Th.7]. Indeed, taking the integral version of the parallel transport equation $\dfrac{\mathbb{D}}{\mathbb{D}z}(\overset{h,y_t^h}{\mathbb{T}}{}_a^z y_{t,h(a)}^{(n)}) = 0$, the expression $\dfrac{\partial}{\partial z}(\overset{h,y_t^h}{\mathbb{T}}{}_a^z y_{t,h(a)}^{(n)})$ is written via the connection terms. The further application of Newton-Leibnitz formula gives the local increments of $\overset{h,y_t^h}{\mathbb{T}}{}_a^z y_{t,h(a)}^{(n)} - y_{t,h(a)}^{(n)}$ as the integrals on $[a,z]$ of these connection terms. Finally, calculating the Stratonovich differential of these integral formulas, comparing them with the representation (33) and proceeding further by scheme [2, (3.11)-(3.19)] the relation (34) is found.

After that the application of (16) to (34) leads to

$$\begin{cases} K_\alpha^{(n)}(z) = \overset{h,y_t^h}{\mathbb{T}}{}_a^z M_\alpha^{(n)} + \\ + \iint\limits_a^z \overset{h,y_t^h}{\mathbb{T}}{}_u^z \left\{ R_{y_t^{h(u)}}\left(\overset{h,y_t^h}{\mathbb{T}}{}_a^u\, y_{t,h(a)}^{(n)}, A_\alpha(y_t^{h(u)})\right) y_{t,h(u)}^{(1)}[h'(u)]\right\} du; \\ \\ L^{(n)}(z) = \overset{h,y_t^h}{\mathbb{T}}{}_a^z N^{(n)} + \\ + \iint\limits_a^z \overset{h,y_t^h}{\mathbb{T}}{}_u^z \left\{ R_{y_t^{h(u)}}\left(\overset{h,y_t^h}{\mathbb{T}}{}_a^u\, y_{t,h(a)}^{(n)}, A_0(y_t^{h(u)})\right) y_{t,h(u)}^{(1)}[h'(u)]\right\} du \end{cases} \tag{35}$$

To obtain relation (19), by schemes of [1] and [7, Sect.4.4-4.5] the following two estimates on the continuity and regularity of variations are required: for any Lipschitz continuous path $h\in Lip([a,b],M)$

$$\mathbf{E}||\, y^{(n)}_{t,h(b)} - \overset{h,y^h_t}{\mathbb{T}}{}^b_a\, y^{(n)}_{t,h(a)}\, ||^p_{T^{1,0}_{y_t^{h(b)}}\otimes T^{0,n}_{h(b)}} \le |b-a|^p ||h'||^p_{L^\infty([a,b],TM)} e^{K_{p,n}t}$$

$$\times pol_{p,n}\left(1+\rho(h(a),o)+|b-a|\cdot||h'||_{L^\infty([a,b],TM)}\right); \qquad (36)$$

$$\mathbf{E}||\, y^{(n)}_{t,h(b)} - \overset{h,y^h_t}{\mathbb{T}}{}^b_a\, y^{(n)}_{t,h(a)} - y^{(n+1)}_{t,h(b)}\left[\int_a^b \overset{h}{\mathbf{T}}{}^b_z\, h'(z)dz\right]\, ||^p_{T^{1,0}_{y_t^{h(b)}}\otimes T^{0,n}_{h(b)}} \le$$

$$\le |b-a|^{2p} ||h'||^{2p}_{L^\infty([a,b],TM)} e^{K_{p,n}t}$$

$$\times pol_{p,n}\left(1+\rho(h(a),o)+|b-a|\cdot||h'||_{L^\infty([a,b],TM)}\right) \qquad (37)$$

with some polynomials $pol_{p,n}(\cdot)$, depending on the order of nonlinearity $\mathbf{k}$ (10), $\overset{h}{\mathbf{T}}{}^b_z$ denoting the classical parallel transport of tensor along path h from $h(z)$ to $h(b)$.

By the theory of absolute continuous functions, estimate (36) leads to the existence of derivative $\frac{\mathbb{D}}{\mathbb{D}z}\overset{h,y^h_t}{\mathbb{T}}{}^b_z y^{(n)}_{t,h(z)}$ and estimate (37) calculates this derivative, leading to (19).

To obtain estimate (36), let us first note, that by (29)

$$M_\gamma{}^m{}_\alpha(b) - \overset{h,y^h_t}{\mathbb{T}}{}^b_a M_\gamma{}^m{}_\alpha(a) = \nabla^y_\ell A^m_\alpha(y_t^{h(b)})[\mathbb{W}_\gamma y^\ell_{t,h(b)}] -$$

$$- \overset{h,y^h_t}{\mathbb{T}}{}^b_a \left(\nabla^y_\ell A^m_\alpha(y_t^{h(a)})[\mathbb{W}_\gamma y^\ell_{t,h(a)}]\right) +$$

$$+ \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\ge 2}\Big\{ K'^{,h(b)}_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}y_{t,h(b)},\ldots,\mathbb{W}_{\beta_s}y_{t,h(b)}) -$$

$$- \overset{h,y^h_t}{\mathbb{T}}{}^b_a\left(K'^{,h(a)}_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}y_{t,h(a)},\ldots,\mathbb{W}_{\beta_s}y_{t,h(a)})\right)\Big\} =$$

$$= \nabla^y_\ell A^m_\alpha(y_t^{h(b)})[\mathbb{W}_\gamma y^\ell_{t,h(b)} - \overset{h,y^h_t}{\mathbb{T}}{}^b_a \mathbb{W}_\gamma y^\ell_{t,h(a)}] +$$

$$+ \left\{\nabla^y_\ell A^m_\alpha(y_t^{h(b)}) - \overset{h,y^h_t}{\mathbb{T}}{}^b_a\left[\nabla^y_\ell A^m_\alpha(y_t^{h(a)})\right]\right\}[\overset{h,y^h_t}{\mathbb{T}}{}^b_a \mathbb{W}_\gamma y^\ell_{t,h(a)}] +$$

$$+\sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2}\left\{K'^{,h(b)}_{\beta_1,\ldots,\beta_s}-\overset{h,y_t^h}{\mathbb{I}}{}^b_a K'^{,h(a)}_{\beta_1,\ldots,\beta_s}\right\}(\mathbb{\nabla}_{\beta_1}y_{t,h(b)},\ldots,\mathbb{\nabla}_{\beta_s}y_{t,h(b)})+$$

$$+\sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2}\sum_{j=1}^{s}[\overset{h,y_t^h}{\mathbb{I}}{}^b_a K'^{,h(a)}_{\beta_1,\ldots,\beta_s}]\Bigg(\overset{h,y_t^h}{\mathbb{I}}{}^b_a\mathbb{\nabla}_{\beta_1}y_{t,h(a)},\ldots,\overset{h,y_t^h}{\mathbb{I}}{}^b_a\mathbb{\nabla}_{\beta_{j-1}}y_{t,h(a)},$$

$$\mathbb{\nabla}_{\beta_j}y_{t,h(b)}-\overset{h,y_t^h}{\mathbb{I}}{}^b_a\mathbb{\nabla}_{\beta_j}y_{t,h(a)},\mathbb{\nabla}_{\beta_{j+1}}y_{t,h(b)},\ldots,\mathbb{\nabla}_{\beta_s}y_{t,h(b)}\Bigg).$$

Due to (16) and the first order regularity of process y_t^x on initial data (18), multiples $\nabla^y_\ell A^m_\alpha(y_t^{h(b)})-\overset{h,y_t^h}{\mathbb{I}}{}^b_a\nabla^y_\ell A^m_\alpha(y_t^{h(a)})$ and $K'^{,h(b)}_{\beta_1,\ldots,\beta_s}-\overset{h,y_t^h}{\mathbb{I}}{}^b_a K'^{,h(a)}_{\beta_1,\ldots,\beta_s}$ are represented as integrals on $[a,b]$ with linear dependence on factor h'. Thus, by equations (22), (29), (33) and (35), the principal parts of equations on the continuity difference $\epsilon_t^{(n)}=y^{(n)}_{t,h(b)}-\overset{h,y_t^h}{\mathbb{I}}{}^b_a y^{(n)}_{t,h(a)}$ has form

$$\begin{aligned}&\delta(\epsilon_t^{(n)})=-\Gamma(\epsilon_t^{(n)},\delta y_t^{(h(b)})+\\&+\textstyle\sum_\alpha\left\{\nabla A_\alpha[\epsilon_t^{(n)}]+P_\alpha^{(n)}(A_\alpha,R,\ \epsilon^{(1)},\ldots,\varepsilon^{(n-1)})\right\}\delta W^\alpha+\\&+\left\{\nabla A_0[\epsilon_t^{(n)}]+P_\alpha^{(n)}(A_\alpha,R,A_0\ \epsilon^{(1)},\ldots,\varepsilon^{(n-1)})\right\}dt,\end{aligned}$$

with linear with respect to factor h' and integral on $[a,b]$ terms $P_\alpha^{(n)},\ P_0^{(n)}$, depending in the polynomial way of coefficients $A_\alpha,\ A_0$, curvature R and their covariant derivatives.

Therefore, proceeding like in the previous part of the proof (30)-(32), singling out the dissipativity condition and using $e_0^{(n)}=0$, the inequality (36) is proved in the inductive on the order of variation way.

Similar, but more bookkeeping arguments work for the differentiability difference $\Delta_t^{(n)}=y^{(n)}_{t,h(b)}-\overset{h,y_t^h}{\mathbb{I}}{}^b_a y^{(n)}_{t,h(a)}-y^{(n+1)}_{t,h(b)}\left[\int_a^b\overset{h}{\mathbf{T}}{}^b_z h'(z)dz\right]$ in (37), however there are applied relation like

$$\begin{aligned}&S_{\beta_1,\ldots,\beta_s}(y_t^{h(b)})-\overset{h,y_t^h}{\mathbb{I}}{}^b_a S_{\beta_1,\ldots,\beta_s}(y_t^{h(b)})-\\&-\mathbb{\nabla}^y S_{\beta_1,\ldots,\beta_s}(y_t^{h(b)})\left[\int_a^b\overset{h,y_t^h}{\mathbb{I}}{}^b_a y^{(1)}_{t,h(z)}[h'(z)]dz\right]=\\&=\int_a^b\mathbb{\nabla}^y S_{\beta_1,\ldots,\beta_s}(y_t^{h(z)})\left[\overset{h,y_t^h}{\mathbb{I}}{}^b_a y^{(1)}_{t,h(z)}[h'(z)]\right]dz-\end{aligned}$$

$$-\mathbb{V}^y S_{\beta_1,...,\beta_s}(y_t^{h(b)})\left[\int_a^b \overset{h,y_t^h}{\mathbb{T}}{}_a^b y^{(1)}_{t,h(z)}[h'(z)]dz\right]=$$
$$=\int_a^b dz\int_z^b du\left(\overset{h,y_t^h}{\mathbb{T}}{}_u^z \mathbb{V}^y\mathbb{V}^y S_{\beta_1,...,\beta_s}(y_t^{h(u)})\right)\times$$
$$\times\left[\overset{h,y_t^h}{\mathbb{T}}{}_u^z\left(y^{(1)}_{t,h(u)}[h'(u)]\right),\ y^{(1)}_{t,h(z)}[h'(z)]\right].$$

to conclude that the differentials of difference expressions have form

$$\delta(\Delta_t^{(n)})=-\Gamma(\epsilon_t^{(n)},\delta y_t^{h(b)})+$$
$$+\sum_\alpha\left\{\nabla A_\alpha[\Delta_t^{(n)}]+Q_\alpha^{(n)}(A_\alpha,R,\ \Delta^{(1)},...,\Delta^{(n-1)})\right\}\delta W^\alpha+$$
$$+\left\{\nabla A_0[\epsilon_t^{(n)}]+Q_\alpha^{(n)}(A_\alpha,R,A_0\ \Delta^{(1)},...,\Delta^{(n-1)})\right\}dt$$

with quadratic with respect to factor h' and integral on $[a,b]^2$ multiples $Q_\alpha^{(n)},\ Q_0^{(n)}$. Due to $\Delta_0^{(n)}=0$ this leads to (37) with powers $2p$ in the r.h.s. ■

3.Proof of C^∞-regularity of semigroup P_t (Theorem 1).

First we are going to obtain the representation formula for derivatives of semigroup via new type variations (7).

Theorem 4. *For any $f\in C^n_{\vec q}(M)$ the semigroup P_tf is n-times continuously differentiable on x for any $t\geq 0$. Its high order derivatives are defined by (7).*

Proof. Introduce notations

$$\delta_m(f,x,t)=\sum_{j_1+...+j_\ell=m,\ell\geq 1}\mathbf{E}\left\langle\nabla^{(\ell)}_{y_t^x}f(y_t^x),\mathbb{V}^{(j_1)}y_t^x\otimes...\otimes\mathbb{V}^{(j_\ell)}y_t^x\right\rangle_{T^{0,\ell}_{y_t^x}M}\tag{38}$$

for the left hand sides of (7). First we are going to demonstrate that for any $f\in C^n_{\vec q}(M)$ expressions $\delta_m(f,x,t)\in T_x^{0,m}M$ are continuous on $x\in M$ for any $m=1,...,n,\ t\geq 0$.

Let $h\in Lip([a,b],M)$ be any Lipschitz path. Let's apply (20) to find majorant function for terms under expectation $\mathbf{E}$ in $[a,b]\ni z\to\delta_m(f,h(z),t)$. From (18) and $||\nabla_x\rho(x,o)||\leq 1$ follows estimate

$$\rho(o,y_t^{h(z)})\leq\rho(o.y_t^{h(a)})+\int_a^z||\nabla_{y_t^{h(\theta)}}\rho(o,y_t^{h(\theta)})||\cdot||\frac{dy_t^{h(\theta)}}{d\theta}||d\theta\leq$$

$$\leq \rho(o, y_t^{h(a)}) + \int_a^b ||\mathbb{V}^{(1)} y_t^{h(\theta)}|| \cdot ||h'(\theta)|| d\theta.$$

Due to $f \in C_{\vec{q}}^n(M)$ it leads to

$$||\nabla^{(\ell)} f(y_t^{h(z)})|| \leq ||f||_{C_{\vec{q}}^n} p_\ell(\rho^2(o, y_t^{h(z)})) \leq$$

$$\leq K_{p_\ell} ||f||_{C_{\vec{q}}^n} \left(1 + \rho(o, y_t^{h(z)})\right)^{2\deg(p_\ell)} \leq$$

$$\leq K_{p_\ell} ||f||_{C_{\vec{q}}^n} \left(1 + \rho(o, y_t^{h(a)}) + ||h'||_{L^\infty[a,b]} \int_a^b ||\mathbb{V}^{(1)} y_t^{h(\theta)}|| d\theta\right)^{2\deg(p_\ell)} \tag{39}$$

and the last expression provides uniform on $z \in [a, b]$ majorant, which is integrable due to estimates (20) and (21).

In a similar way we find majorant for variational processes in expression $\delta_m(f, h(z), t),\ z \in [a, b]$. Due to (19)

$$\begin{aligned} &\forall\, z \in [a, b] \quad ||\, \mathbb{V}^{(j)} y_t^{h(z)} ||_{y_t^{h(z)}} \leq \\ &\leq ||\, \mathbb{V}^{(j)} y_t^{h(a)} ||_{y_t^{h(a)}} + ||h'||_{L^\infty[a,b]} \textstyle\int_a^b ||\, \mathbb{V}^{(\ell+1)} y_t^{h(\theta)} ||_{y_t^{h(\theta)}}\, d\theta \end{aligned} \tag{40}$$

and the right hand side of (40) is integrable in any power due to (20).

Property (19) and majorants (39),(40) lead to a.e. continuity on parameter $z \in [a, b]$ of expressions under expectation $\mathbf{E}$ in $\delta_m(f, h(z), t),\ m = 0, ..., n$ for $f \in C_{\vec{q}}^n(M)$. The further application of Lebesgue majorant theorem demonstrates the continuity of mappings

$$[a, b] \ni z \longrightarrow \delta_m(f, h(z), t), \quad m = 0, ..., n,$$

for any Lipschitz path $h \in Lip([a, b], M)$ and $f \in C_{\vec{q}}^n(M)$.

Since such continuity along paths h represents one of possible characterizations of continuous mappings, we conclude the a.e. continuity of expressions δ_m

$$\text{mapping} \ \ M \ni x \rightarrow \delta_m(f, x, t) \in T^{0,m} M \ \ \text{is continuous}$$

for any $f \in C_{\vec{q}}^n(M)$ and $t \geq 0,\ m = 0, ..., n$.

Now we can recurrently prove the required relation $\nabla^{(m)} P_t f(x) = \delta_m(f, x, t)$.

Base of recurrence ($m = 1$). Using representation $P_t f(x) = \mathbf{E} f(y_t^x)$ and (41) for $\ell = 0$ we obtain

$$P_t f(h(b)) - P_t f(h(a)) = \mathbf{E}\left[f(y_t^{h(b)}) - f(y_t^{h(a)})\right] =$$

$$= \mathbf{E} \int_a^b < \nabla f(y_t^{h(z)}), \mathbb{V}^{(1)} y_t^{h(z)} [h'(z)] > dz.$$

Due to the existence of majorants (39) and (40) for $\ell = 1$, the expectation and integral can be changed in order. We obtain that for any $h \in Lip([a,b], M)$

$$P_t f(h(b)) - P_t f(h(a)) = \int_a^b \mathbf{E} < \nabla f(y_t^{h(z)}), \mathbb{V}^{(1)} y_t^{h(z)} [h'(z)] > dz$$

and by the theory of absolutely continuous functions conclude the existence of derivative

$$\frac{dP_t f(h(z))}{dz} = \mathbf{E}\langle \nabla f(y_t^{h(z)}), \mathbb{V}^{(1)} y_t^{h(z)} [h'(z)]\rangle = \langle \delta_1(f, h(z), t), h'(z)\rangle.$$

Since $\delta_1(f,x,t)$ is continuous on x, this leads to the existence of continuous first order derivative $\nabla P_t f(x)$ and identity $\nabla_x P_t f(x) = \delta_1(f,x,t)$.

Recurrence step. Suppose that we already proved relation $\nabla_x^{(\ell)} P_t f(x) = \delta_\ell(f,x,t)$ for any $\ell = 0, ..., m < n$. Let us show it for $m+1$.

First note that from property $\frac{dy_t^{h(z)}}{dz} = \mathbb{V}^{(1)} y_t^{h(z)}[h'(z)]$ (18) and a.e. relations (19) follows a.e. relation

$$\begin{aligned} &\forall\, \ell = \overline{0, n-1} \quad \nabla^{(\ell)} f(y_t^{h(b)}) - \overset{h,y_t^h}{\mathbf{I\!T}}{}_a^b \left[\nabla^{(\ell)} f(y_t^{h(a)})\right] = \\ &= \int_a^b \overset{h,y_t^h}{\mathbf{I\!T}}{}_z^b \left(\nabla^{(\ell+1)} f(y_t^{h(z)}) \left[\mathbb{V}^{(1)} y_t^{h(z)}[h'(z)]\right]\right) dz \end{aligned} \tag{41}$$

for any $f \in C_0^n(M)$. Taking cutoffs $f\chi_U$ with $\chi_U|_U = 1$, $\chi_U \in C_0^\infty(M, [0,1])$ and tending $U \nearrow M$, representation (41) can be closed to any $f \in C_{\overline{q}}^n(M)$.

Consider the corresponding difference

$$\nabla^{(m)} P_t f(h(b)) - \overset{h}{\mathbf{T}}{}_a^b \left[\nabla^{(m)} P_t f(h(a))\right] =$$

$$= \mathbf{E} \sum_{j_1 + ... + j_\ell = m,\ \ell \geq 1} \Bigg[\langle \nabla^{(\ell)} f(y_t^{h(b)}), [\mathbb{V}^{(j_1)} y_t^{h(b)}] \otimes ... \otimes [\mathbb{V}^{(j_\ell)} y_t^{h(b)}]\rangle_{T^{0,\ell}_{y_t^{h(b)}}} -$$

$$- \overset{h}{\mathbf{T}}{}_a^b \langle \nabla^{(\ell)} f(y_t^{h(a)}), [\mathbb{V}^{(j_1)} y_t^{h(a)}] \otimes ... \otimes [\mathbb{V}^{(j_\ell)} y_t^{h(a)}]\rangle_{T^{0,\ell}_{y_t^{h(a)}}} \Bigg].$$

Relations (41) and (19) lead to

$$\sum_{j_1+\ldots+j_\ell=m,\ \ell\geq 1}\Big[\langle\nabla^{(\ell)}f(y_t^{h(b)}),[\mathbb{V}^{(j_1)}y_t^{h(b)}]\otimes\ldots\otimes[\mathbb{V}^{(j_\ell)}y_t^{h(b)}]\rangle-$$

$$-\overset{h}{\mathbf{T}}{}_a^b\langle\nabla^{(\ell)}f(y_t^{h(a)}),[\mathbb{V}^{(j_1)}y_t^{h(a)}]\otimes\ldots\otimes[\mathbb{V}^{(j_\ell)}y_t^{h(a)}]\rangle\Big]=$$

$$=\int_a^b\sum_{j_1+\ldots+j_\ell=m+1,\ \ell\geq 1}\overset{h}{\mathbf{T}}{}_z^b\Big[\langle\nabla^{(\ell)}f(y_t^{h(z)}),[\mathbb{V}^{(j_1)}y_t^{h(z)}]\otimes\ldots$$

$$\otimes[\mathbb{V}^{(j_\ell)}y_t^{h(z)}]\rangle[h'(z)]\Big]\,dz,$$

i.e. recover the structure of integrand in (38).

The existence of majorants (39) and (40) permits to change the order of integration and expectation, leading to

$$\nabla^{(m)}P_tf(h(b))-\overset{h}{\mathbf{T}}{}_a^b\left[\nabla^{(m)}P_tf(h(a))\right]=\int_a^b\overset{h}{\mathbf{T}}{}_z^b\left[\delta_{m+1}(f,h(z),t)[h'(z)]\right]dz.$$

Therefore the mapping $[a,b]\ni z\to\overset{h}{\mathbf{T}}{}_z^b\left[\nabla^{(m)}P_tf(h(z))\right]$ is absolutely continuous with derivative

$$\frac{d\,\overset{h}{\mathbf{T}}{}_z^b\left[\nabla^{(m)}P_tf(h(z))\right]}{dz}=\overset{h}{\mathbf{T}}{}_z^b\left[\delta_{m+1}(f,h(z),t)[h'(z)]\right].$$

Since $\delta_{m+1}(f,x,t)$ is continuous on x, we conclude that the $(m+1)^{th}$ derivative of semigroup is represented by $\delta_{m+1}(f,x,t)$. ■

The final step of the proof of Theorem 4 lies in the verification of estimate (13). It follows the scheme of [2, Th.15] with application of estimates (20) instead of nonlinear estimates on variations.

Theorem 5. *Under conditions of Theorem 1 estimate (13) holds.*

Proof. We apply (20) and (21) to estimate the corresponding seminorms

$$\frac{\|(\nabla^x)^i P_t f(x)\|_{T_x^{(0,i)}}}{q_i(\rho^2(x,o))} \le$$

$$\le \sum_{j_1+\ldots+j_\ell,\ \ell\ge 1} \frac{\|\mathbf{E}\,\langle (\nabla^y)^\ell f(y_t^x)\,,\,\mathbb{V}^{(j_1)}y_t^x \otimes \ldots \otimes \mathbb{V}^{(j_\ell)}y_t^x\rangle_{T_y^{(0,i)}}\|_{T_x^{(0,i)}}}{q_i(\rho^2(x,o))} \le$$

$$\le \sum_{j_1+\ldots+j_\ell,\ \ell\ge 1} \left(\sup_{y_t^x\in M} \frac{\|(\nabla^y)^\ell f(y_t^x)\|_{T_y^{(0,\ell)}}}{q_\ell(\rho^2(y_t^x,o))} \right) \times$$

$$\times \frac{\mathbf{E} q_\ell(\rho^2(y_t^x,o)) \|\mathbb{V}^{(j_1)}y_t^x\| \ldots \|\mathbb{V}^{(j_\ell)}y_t^x\|}{q_i(\rho^2(x,o))} \le \|f\|_{C^n_{\vec q}} \times$$

$$\times \sum_{j_1+\ldots+j_\ell,\ \ell\ge 1} \frac{\left(\mathbf{E} q_\ell^{\ell+1}(\rho^2(y_t^x,o))\right)^{1/(\ell+1)} \prod_{m=1}^{\ell} \left(\mathbf{E}\|\mathbb{V}^{(j_m)}y_t^x\|^{\ell+1}\right)^{1/(\ell+1)}}{q_i(\rho^2(x,o))} \le$$

$$\le K^2 e^{M't}\|f\|_{C^n_{\vec q}} \sum_{j_1+\ldots+j_\ell,\ \ell\ge 1} \frac{q_\ell(\rho^2(x,o)) \prod_{m=1}^{\ell} (1+\rho^2(x,o))^{\mathbf{k}(j_m-1)/2}}{q_i(\rho^2(x,o))} \le$$

$$\le K^2 e^{M't}\|f\|_{C^n_{\vec q}} \sum_{j_1+\ldots+j_\ell,\ \ell\ge 1} \frac{q_\ell(\rho^2(x,o))(1+\rho^2(x,o))^{\mathbf{k}(i-\ell)/2}}{q_i(\rho^2(x,o))},$$

leading to hierarchy (11). Above we also applied that for $q_i \ge 1$ of polynomial behaviour there is K such that $\frac{1}{K}(1+b)^{deg(q_i)} \le q_i(b) \le K(1+b)^{deg(q_i)}$, so from (21) follows

$$\mathbf{E}\left[q_i(\rho^2(o,y_t^x))\right]^n \le K^n \mathbf{E}\left[1+\rho^2(o,y_t^x)\right]^{n\cdot deg(q_i)} \le$$

$$\le K^n e^{n\cdot deg(q_i)Mt}\left[1+\rho^2(o.x)\right]^{n\cdot deg(q_i)} \le K^{2n} e^{n\cdot deg(q_i)Mt} q_i(\rho^2(o,x)). \quad \blacksquare$$

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