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Weak solutions to one initial-boundary value problem with three boundary conditions for quasilinear evolution equations of the third order

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Abstract. Global well-posedness in a class of weak solutions is established to one initial-boundary value problem with three boundary conditions for a wide class of quasilinear dispersive evolution equations of the third order in the multidimensional case. The considered class of equations generalizes the Korteweg–de Vries, the Korteweg–de Vries–Burgers and the Zakharov–Kuznetsov equations.

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The paper is concerned with global well-posedness of one nonhomogeneous initial-boundary value problem for quasilinear partial differential equations of the type

$$u_t - P(\partial_x)u + \operatorname{div}_x g(u) = f(t,x), \tag{1}$$

where $u = u(t,x)$, $t \geq 0$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, $g = (g_1, \dots, g_n)$,

$$\operatorname{div}_x g(u) = \sum_{j=1}^{n} g_j'(u) u_{x_j}, \qquad P(\partial_x) = \sum_{|\alpha| \leq 3} a_\alpha \partial_x^\alpha$$

is a linear differential operator with constant real coefficients a_α, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

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These equations are studied in the domain $Q_T = (0,T) \times \Omega$, $T > 0$ – arbitrary, $\Omega = \{x \in \mathbb{R}^n : x_n \in (0,1)\} = \mathbb{R}^{n-1} \times (0,1)$. Initial condition

$$u\big|_{t=0} = u_0(x), \quad x \in \Omega, \tag{2}$$

and three boundary conditions

$$\begin{aligned} u\big|_{x_n=0} &= u_1(t,x'), \\ u\big|_{x_n=1} &= u_2(t,x'), \qquad\qquad (t,x') \in S_T, \\ u_{x_n}\big|_{x_n=1} &= u_3(t,x'), \end{aligned} \tag{3}$$

where $x' = (x_1, \dots, x_{n-1})$, $S_T = (0,T) \times \mathbb{R}^{n-1}$, are set.

The equations (1) generalize well-known equations, describing propagation of nonlinear waves in dispersive media, namely, the Korteweg – de Vries equation

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{R}, \tag{4}$$

the Korteweg – de Vries – Burgers equation

$$u_t + u_{xxx} - \delta u_{xx} + uu_x = 0, \quad x \in \mathbb{R}, \quad \delta > 0, \tag{5}$$

the Zakharov – Kuznetsov equation in two and three spatial dimensions

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0, \quad (x,y) \in \mathbb{R}^2, \tag{6}$$

$$u_t + u_{xxx} + (u_{yy} + u_{zz})_x + uu_x = 0, \quad (x,y,z) \in \mathbb{R}^3. \tag{7}$$

The initial value problem for equations of the (1) type (in fact, of an arbitrary high odd order) in the multimensional case was previously studied in [15] and [4]. An initial-boundary value problem for the equation (1) in a domain $\Pi_T^+ = (0,T) \times \{x \in \mathbb{R}^n : x_n > 0\}$ with one boundary condition u_1 was considered in [6] (P was assumed to be a homogeneous operator of the third order). The Zakharov–Kuznetsov equation (6) was studied in [5, 7, 13, 14].

Global theory for quasilinear dispersive equations is based on conservation laws, where the first one is in L_2. Indeed, if u is a smooth and decaying at infinity solution to the initial value problem for any of the equations (4)–(7), then for $t > 0$

$$\|u(t,\cdot)\|_{L_2} \le \|u_0\|_{L_2}. \tag{8}$$

But this estimate is not sufficient to construct global solution for $u_0 \in L_2$ because of nonlinearity.

Additional estimate can be obtained via the so called local smoothing effect. It is based on the following simple property of the third derivative:

$$2\int_I u_{xxx} u\rho\, dx = 3\int_I u_x^2\rho'\, dx - \int_I u^2\rho''' dx - u_x^2\rho\big|_{\partial I}, \tag{9}$$

where ρ is a smooth, bounded, increasing function on a certain interval (bounded or unbounded) $I \subset \mathbb{R}$ and $u|_{\partial I} = 0$. The presence of the definite term $u_x^2\rho'$ provides extra smoothing of a solution to (4) in comparison with initial profile $u_0 \in L_2$ on any bounded subinterval of I, [9,12]. When I is bounded the local smoothing effect transforms to the global one on the whole interval I, [2,11,14].

In order to ensure properties similar to (8), (9) the operator P in the present paper is subjected to the following assumptions. Consider the representation

$$P(\partial_x) = \sum_{k=0}^{3} \sum_{|\alpha|=k} a_\alpha \partial_x^\alpha \equiv \sum_{k=0}^{3} P_k(\partial_x). \tag{10}$$

Let

$$P_k(\xi) \equiv \sum_{|\alpha|=k} a_\alpha \xi^\alpha, \quad \xi = (\xi_1, \dots, \xi_n), \quad \xi^\alpha = \xi^{\alpha_1} \dots \xi^{\alpha_n},$$

be the symbol of the operator P_k. Assume that

$$1)\ Q_2(\xi) \equiv \frac{\partial}{\partial \xi_n} P_3(\xi) < 0 \quad \forall\, \xi \neq 0, \qquad 2)\ P_2(\xi) \geq 0 \quad \forall\, \xi \in \mathbb{R}^n. \tag{11}$$

It follows from the first condition (11), that there exists a constant $c > 0$ such that $Q_2(\xi) \leq -c|\xi|^2\ \forall\, \xi \in \mathbb{R}^n$, therefore, the differential operator $Q_2(\partial_x)$ with the symbol $Q_2(\xi)$ is elliptic.

Note also, that if one represents P_k in such a way:

$$P_k(\partial_x) \equiv a_k \partial_{x_n}^k + \sum_{l=1}^{k} P_{kl}(\partial_{x'}) \partial_{x_n}^{k-l}, \tag{12}$$

where $P_{kl}(\partial_{x'})$ are homogeneous differential operators of the orders l and $a_k = a_{(0,\dots,0,k)}$, then (11) yields that

$$a_3 < 0. \tag{13}$$

The first condition (11) means, that the operator $\partial_t - P_3(\partial_x)$ is 3-hyperbolic in the direction x_n, [16].

In fact, the second condition (11) provides a global estimate on solutions of the (8) type, while the first one — a local smoothing effect. For the first time the first condition (11) for equations of the (1) type appeared in [4]. Other conditions, that provide local smothing effect for dispersive equations, can be found in [10, 13, 15].

The next lemma is concerned with properties of the operator P (compare with (9)).

Lemma 1. *Let the inequalities* (11) *be satisfied. Then there exist two positive constants* c_0 *and* c_1 *such that for any function* $\varphi(x) \in H^3(\Omega)$, $\varphi|_{x_n=0} = \varphi|_{x_n=1} = 0$,

$$\int\limits_{\Omega} P(\partial_x)\varphi \cdot \varphi\rho(x_n)\,dx \le \frac{a_3}{2} \int\limits_{\mathbb{R}^{n-1}} \bigl(\varphi_{x_n}^2\big|_{x_n=0}\rho(0) - \varphi_{x_n}^2\big|_{x_n=1}\rho(1)\bigr)\,dx' \\ - c_0 \int\limits_{\Omega} |\operatorname{grad}_x \varphi|^2 \rho'(x_n)\,dx + c_1 \int\limits_{\Omega} \varphi^2\rho(x_n)\,dx, \quad (14)$$

where either $\rho(x_n) \equiv 1$ *or* $\rho(x_n) \equiv 1 + x_n$ *and* a_3 *is the negative constant from* (12).

Proof. For $\rho \equiv 1$ the inequality (14) is obvious. For $\rho(x_n) \equiv 1 + x_n$ by virtue of the Leibniz formula write the equality

$$P(\partial_x)\varphi \cdot \varphi\rho(x_n) = P(\partial_x)(\varphi\rho^{1/2}) \cdot \varphi\rho^{1/2} - \frac{1}{2}Q_2(\partial_x)\varphi \cdot \varphi \\ + \sum_{j=0}^{1} Q_j(\partial_x)\varphi \cdot \varphi(\rho^{1/2})^{(3-j)}\rho^{1/2},$$

where $Q_j(\partial_x)$, $j = 0$ and 1, are certain homogeneous linear differential operators of the orders j. Then (14) in this case succeeds from (14) for $\rho \equiv 1$ and the ellipticity of the operator Q_2. $\square$

Consider the formally adjoint to P operator

$$P^*(\partial_x) = \sum_{k=0}^{3} (-1)^k P_k(\partial_x).$$

Lemma 2. *Let the inequalities* (11) *be satisfied. Then for any function* $\varphi(x) \in H^3(\Omega)$, $\varphi|_{x_n=0} = \varphi|_{x_n=1} = 0$,

$$\int\limits_{\Omega} P^*(\partial_x)(\varphi\rho(x_n))\varphi\,dx \le \frac{a_3}{2} \int\limits_{\mathbb{R}^{n-1}} \bigl(\varphi_{x_n}^2\big|_{x_n=1}\rho(1) - \varphi_{x_n}^2\big|_{x_n=0}\rho(0)\bigr)\,dx' \\ - c_0 \int\limits_{\Omega} |\operatorname{grad}_x \varphi|^2 \rho'(x_n)\,dx + c_1 \int\limits_{\Omega} \varphi^2\rho(x_n)\,dx, \quad (15)$$

where either $\rho(x_n) \equiv 1$ *or* $\rho(x_n) \equiv 1 + x_n$ *and* c_0, c_1 *are the constants from* (14).

Proof. The inequality (15) follows from (14) and the equality

$$\int\limits_{\Omega} P^*(\partial_x)(\varphi\rho(x_n))\varphi\,dx = \int\limits_{\Omega} P(\partial_x)\varphi\cdot\varphi\rho(x_n)\,dx \\ + a_3 \int\limits_{\mathbb{R}^{n-1}} \left(\varphi^2_{x_n}\big|_{x_n=1}\rho(1) - \varphi^2_{x_n}\big|_{x_n=0}\rho(0)\right) dx'.$$

□

Any function $g_j(u)$, $j = 1,\dots,n$, is assumed to be in the class $C^1(\mathbb{R})$ and to satisfy for certain constants $0 \le b_j \le 1$ and $\widetilde{c} > 0$ the following inequality:

$$|g'_j(u)| \le \widetilde{c}(|u|^{b_j} + 1) \quad \forall\, u \in \mathbb{R}, \tag{16}$$

so these functions have at most quadratic rate of growth (additional restrictions on b_j are specified further). Without loss of generality we also assume that $g_j(0) = 0$.

If B is a certain Banach space, the symbol $L_p(0,T;B)$ is used in the conventional sense of Bochner measurable mappings from the interval $(0,T)$ into B, summable with the degree p (essentially bounded if $p = +\infty$). The symbols $C([0,T];B)$ and $C_w([0,T];B)$ denote respectively the spaces of continuous and weakly continuous mappings from $[0,T]$ into B. It is known, that $C_w([0,T];B) \subset L_\infty(0,T;B)$, [8].

We use the well-known interpolational inequality (see, e.g., [1]) for functions $\varphi \in H^1(\Omega)$

$$\|\varphi\|_{L_p(\Omega)} \le c(p)\left(\|\operatorname{grad}\varphi\|^s_{L_2(\Omega)}\|\varphi\|^{1-s}_{L_2(\Omega)} + \|\varphi\|_{L_2(\Omega)}\right), \tag{17}$$

where $2 \le p < +\infty$ for $n = 2$, $2 \le p \le 2n/(n-2)$ for $n > 2$, $s = n/2 - n/p$.

We also use the anysotropic Sobolev spaces for integer $l, m \ge 0$ in such a form:

$$H^{l,m}(\Omega) = \{\varphi(x) : \partial^{\alpha'}_{x'}\varphi \in L_2(\Omega), \partial^{\alpha_n}_{x_n}\varphi \in L_2(\Omega),\ |\alpha'| \le l, \alpha_n \le m\},$$

where multi-index $\alpha' = (\alpha_1,\dots,\alpha_{n-1})$.

Now we can introduce a definition of a weak solution to the considered problem. Let $u_0 \in L_2(\Omega)$, $u_j \in L_2(S_T)$ for $j = 1,2,3$, $f \in L_1(0,T;L_2(\Omega)) + L_2(0,T;H^{-1}(\Omega))$.

Definition 1. *A function $u \in L_2(Q_T)$ is called a weak solution to the problem (1)–(3), if for any function $\phi(t,x) \in L_2(0,T;H^{n+1,3}(\Omega)) \cap H^1(Q_T)$ such, that $\phi|_{t=T} = 0$, $\phi|_{x_n=0} = \phi_{x_n}|_{x_n=0} = \phi|_{x_n=1} = 0$, the functions $g_j(u(t,x))\phi_{x_j} \in L_1(Q_T)$ $\forall\, j$ and*

$$\iint\limits_{Q_T} \Big[u(\phi_t + P^*(\partial_x)\phi) + \sum_{j=1}^{n} g_j(u)\phi_{x_j} \Big]\, dx\, dt + \int\limits_0^T \langle f(t,\cdot), \phi(t,\cdot)\rangle\, dt \\ + \int\limits_\Omega u_0 \phi\big|_{t=0}\, dx + \iint\limits_{S_T} u_2 \big(P_{31}(\partial_{x'}) - a_2\big)\phi_{x_n}\big|_{x_n=1}\, dx'dt \\ + a_3 \iint\limits_{S_T} \big(u_2 \phi_{x_n x_n}\big|_{x_n=1} - u_1 \phi_{x_n x_n}\big|_{x_n=0} - u_3\phi_{x_n}\big|_{x_n=1}\big)\, dx'dt = 0. \quad (18)$$

In fact, solutions to the considered problem are constructed in a more smooth class

$$X(Q_T) = C_w([0,T]; L_2(\Omega)) \cap L_2(0,T;H^1(\Omega)).$$

We also use an auxiliary space

$$\widetilde{X}(Q_T) = C([0,T]; L_2(\Omega)) \cap L_2(0,T;H^1(\Omega)).$$

Remark 1. It is known (see, e.g., [1]) that $H^{n+1,3}(\Omega) \subset W^1_\infty(\Omega)$. Thus, under the condition (16) $(g(u), \operatorname{grad}_x \phi) \in L_1(Q_T)$ for $u \in X(Q_T)$. Moreover, $H^{n+1,3}(\Omega) \subset H^3(\Omega)$ and, in particular, $P^*(\partial_x)\phi \in L_2(Q_T)$.

In order to describe properties of the input data we introduce a special space.

Definition 2. *For certain $T > 0$ let M_T be a space of ordered assemblies (u_0, u_1, u_2, u_3, f) such that*

$$u_0 \in L_2(\Omega), \quad u_1, u_2 \in L_1(0,T;H^3(\mathbb{R}^{n-1})) \cap L_2(0,T;H^1(\mathbb{R}^{n-1})), \\ u_{1t}, u_{2t} \in L_1(0,T;L_2(\mathbb{R}^{n-1})), \quad u_3 \in L_2(S_T), \quad f \in L_1(0,T;L_2(\Omega)),$$

supplied with the natural norm.

Let $\eta(\theta)$ be a certain "cut-off"function, i.e. $\eta \in C^\infty(\mathbb{R})$, $\eta(\theta) \geq 0$, $\eta'(\theta) \geq 0$ for $\theta \in \mathbb{R}$, $\eta(\theta) = 0$ for $\theta \leq 0$, $\eta(\theta) = 1$ for $\theta \geq 1$, $\eta(\theta) + \eta(1-\theta) \equiv 1$.

Along with the problem (1)–(3) we consider an auxiliary initial-boundary value problem

$$v_t - P(\partial_x)v + \operatorname{div}_x g(v+\psi) = F, \quad (t,x)\in Q_T, \tag{19}$$
$$v\big|_{t=0} = v_0, \quad x\in\Omega, \tag{20}$$
$$v\big|_{x_n=0} = v\big|_{x_n=1} = 0, \quad v_{x_n}\big|_{x_n=1} = u_3, \quad (t,x')\in S_T. \tag{21}$$

Let

$$\psi(t,x) \equiv u_1(t,x')\eta(1-x_n) + u_2(t,x')\eta(x_n), \tag{22}$$
$$v_0(x) \equiv u_0(x) - \psi(0,x), \quad F(t,x) \equiv f(t,x) + P(\partial_x)\psi(t,x) - \psi_t(t,x). \tag{23}$$

It is obvious, that under (22), (23) a function

$$v(t,x) \equiv u(t,x) - \psi(t,x) \tag{24}$$

is a solution to the problem (19)–(21) if and only if $u(t,x)$ is a solution to the original problem.

First consider a linear version of (19)–(21).

Lemma 3. *Let $g_j \equiv 0$ $\forall j$, $v_0 \in L_2(\Omega)$, $u_3 \in L_2(S_T)$ for some $T>0$, $F \equiv F_1 + F_2$, where $F_1 \in L_1(0,T;L_2(\Omega))$, $F_2 \in L_2(0,T;H^{-1}(\Omega))$, and the conditions* (11) *be satisfied. Then there exists a solution $v(t,x)$ to the problem* (19)–(21) *in the class $\widetilde{X}(Q_T)$. This solution is unique in $L_2(Q_T)$. For any $t\in(0,T]$*

$$\|v\|_{\widetilde{X}(Q_t)} \le c(T)\big(\|v_0\|_{L_2(\Omega)} + \|u_3\|_{L_2(S_T)} + \|F_1\|_{L_1(0,t;L_2(\Omega))} + \|F_2\|_{L_2(0,t;H^{-1}(\Omega))}\big) \tag{25}$$

and

$$\int\limits_\Omega v^2(t,x)\rho(x_n)\,dx + c_0\int\limits_0^t\int\limits_\Omega |\operatorname{grad}_x v|^2\rho'(x_n)\,dx\,d\tau \le \int_\Omega v_0^2\rho(x_n)\,dx + c\int\limits_0^t\int\limits_\Omega v^2\rho(x_n)\,dx\,d\tau + c\iint\limits_{S_t} u_3^2\,dx'd\tau + 2\int\limits_0^t \langle F(\tau,\cdot), v(\tau,\cdot)\rho(x_n)\rangle\,d\tau, \tag{26}$$

where either $\rho\equiv 1$ or $\rho(x_n)\equiv 1+x_n$.

Proof. We construct a solution via the Galerkin method. Let $\{\varphi_j(x), j = 1, 2, \dots\}$ be a set of linearly independent functions complete in the space $\{\varphi \in H^3(\Omega) : \varphi|_{x_n=0} = \varphi_{x_n}|_{x_n=0} = \varphi|_{x_n=1} = 0\}$ and orthonormal in $L_2(\Omega)$. We seek an approximate solution in the form

$$v_m(t,x) = \sum_{j=1}^{m} c_{mj}(t)\varphi_j(x)$$

via the conditions for $l = 1, 2, \dots, m$

$$\int_\Omega \big[v_{mt} \cdot (1+x_n)\varphi_l - v_m P^*(\partial_x)((1+x_n)\varphi_l)\big]\,dx - \langle F(t,\cdot), (1+x_n)\varphi_l\rangle \\ + 2a_3 \int_{\mathbb{R}^{n-1}} u_3 \varphi_{lx_n}\big|_{x_n=1}\,dx' = 0, \quad t \in [0,T], \tag{27}$$

$$\int_\Omega (v_m\big|_{t=0} - v_0)\varphi_l\,dx = 0. \tag{28}$$

Multiplying (27) by $2c_{ml}(t)$ and summing with respect to l we find that by virtue of (15)

$$\frac{d}{dt}\int_\Omega (1+x_n)v_m^2\,dx + 2c_0\int_\Omega |\operatorname{grad}_x v_m|^2\,dx - 2a_3\int_{\mathbb{R}^{n-1}} v_{mx_n}^2\big|_{x_n=1}\,dx' \\ \le 2c_1\int_\Omega (1+x_n)v_m^2\,dx - 4a_3\int_{\mathbb{R}^{n-1}} u_3 v_{mx_n}\big|_{x_n=1}\,dx' \\ + 2\int_\Omega F_1 \cdot (1+x_n)v_m\,dx + c_0\|(1+x_n)v_m\|^2_{H_0^1(\Omega)} + 4c_0^{-1}\|F_2\|^2_{H^{-1}(\Omega)}. \tag{29}$$

Since $v_m(0,\cdot) \to v_0$ in $L_2(\Omega)$ it follows from (29), that the set $\{v_m\}$ is bounded in the space $L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega))$ and, therefore, a solution $v(t,x)$ to the considered problem can be obtained as a weak limit in this space of a certain subsequence $\{v_{m'}\}$. Moreover, $v \in X(Q_T)$ and the inequalities (25) (where $\widetilde{X}$ is temporarily substituted by X) and (26) for $\rho \equiv 1 + x_n$ are valid.

In order to prove uniqueness we use the Holmgren method. It is easy to see, that after corresponding change of variables the adjoint problem coincides with the original one, so, of course, the desired uniqueness can be obtained if one constructs a solution to the considered problem for $F \in C_0^\infty(Q_T)$, $v_0 \equiv 0$, $u_3 \equiv 0$ in the class $L_2(0,T;H^{n+1,3}(\Omega)) \cap H^1(Q_T)$.

But in this paper we prove existence of such solutions not to this problem, but to a certain approximate one.

Consider first the linear problem of the (19)–(21) type with zero initial and boundary data, where $F \in C_0^\infty(Q_T)$ is substituted by F_t. It is already proved, that a solution $w(t,x)$ to this problem in the class $X(Q_T)$ exists. Let

$$v(t,x) \equiv \int_0^t w(\tau,x)\,d\tau. \tag{30}$$

Then the function v is a solution to the linear problem (19)–(21) (with zero initial and boundary data) and $v \in H^1(Q_T)$. Write down the corresponding integral equality (18) for the function v, where the variables x' are substituted by y', and choose a function ϕ in a special form:

$$\phi(t,y',x_n) = \nu(t)\lambda_h(x'-y')\varphi(x_n),$$

where ν is an arbitrary function from $H^1(0,T)$, $\nu(T) = 0$; λ_h is an averaging kernel with a parameter $h > 0$, i.e. $\lambda_h(y') \equiv h^{1-n}\lambda(y'/h)$, $\lambda \in C_0^\infty(\mathbb{R}^{n-1})$, $\lambda \geq 0$, $\lambda(y') = 0$ if $|y'| \geq 1$, $\int_{\mathbb{R}^{n-1}} \lambda(y')\,dy' = 1$; x' is an arbitrary point in $\mathbb{R}^{n-1}$; $\varphi(x_n)$ is an arbitrary function from $H^3(0,1)$ such that $\varphi(0) = \varphi'(0) = \varphi(1) = 0$. Denote by

$$v_h(t,x) \equiv \int_{\mathbb{R}^{n-1}} \lambda_h(x'-y')v(t,y',x_n)\,dy'$$

an average function to v (where the averaging is performed only with respect to x'; the similar notation is used for F_h), then it follows from (18), that the function v_h is a solution to the linear problem of the (19)–(21) type with zero initial and boundary conditions, where F is substituted by F_h, and $\partial_{x'}^{\alpha'} v_h \in H^1(Q_T)$ for any multi-index α'.

The third derivative with respect to x_n can be expressed from the linear equation (19) itself by virtue of (10) and (12):

$$a_3\partial_{x_n}^3 v_h = v_{ht} - F_h - \sum_{l=1}^{3} P_{3l}(\partial_{x'})\partial_{x_n}^{3-l} v_h - \sum_{k=0}^{2} P_k(\partial_x) v_h. \tag{31}$$

Any derivative with respect to x' of the right part of this equality belongs to the space $L_2(0,T;H^{-1}(\Omega))$, therefore, $\partial_{x'}^{\alpha'} v_h \in L_2(0,T;H^2(\Omega))$ for any multi-index α'. Applying (31) once more we derive that $\partial_{x'}^{\alpha'} v_h \in L_2(0,T;H^3(\Omega))$.

As a result, for any $F \in C_0^\infty(Q_T)$ and $h > 0$ a solution to the linear problem (19)–(21) with zero and boundary data and F substituted by

F_h in the class $L_2(0,T;H^{n+1,3}(\Omega))\cap H^1(Q_T)$ is constructed. Thus, if v is a solution in the space $L_2(Q_T)$ to the linear problem (19)–(21) with zero initial and boundary conditions and zero right part, then for any function $F\in C_0^\infty(Q_T)$ and $h>0$ choosing in (18) $\phi(t,x)\equiv v_h(T-t,-x',1-x_n)$, where v_h is a corresponding solution to the linear problem of the (19)–(21) type for $v_0\equiv 0$, $u_3\equiv 0$ and F substituted by $-F_h(T-t,-x',1-x_n)$, we obtain the equality

$$\iint\limits_{Q_T} vF_h\,dx\,dt=0,$$

whence passing to the limit as $h\to 0$ derive that $v\equiv 0$.

Next we construct more smooth solutions to the considered problem. Assume temporarily that $\partial_{x'}^{\alpha'}v_0\in H_0^3(\Omega)$, $\partial_{x'}^{\alpha'}F,\partial_{x'}^{\alpha'}F_t\in L_2(Q_T)$, $\partial_{x'}^{\alpha'}u_3\in H_0^1(S_T)$ for any multi-index α'. As in the proof of uniqueness consider first a linear problem of the (19)–(21) type, where v_0, u_3, F are substituted by $\Phi\equiv F|_{t=0}+P(\partial_x)v_0$, u_{3t}, F_t. Let $w(t,x)$ be a solution to this problem in the class $X(Q_T)$.

Consider for $j=1,\dots,n-1$ and $h\neq 0$ the function $w_j^h\equiv \frac{1}{h}(w(t,x+he_j)-w(t,x))$, where e_j is the unit vector in the direction x_j. This function is a weak solution to the corresponding linear problem of the (19)–(21) type and we can write down the analogue of the inequality (25), where v, v_0, u_3, F are substituted by w_j^h, Φ_j^h, u_{3tj}^h, F_{tj}^h (the notations Φ_j^h etc. are the same as w_j^h, the space X is used instead of $\widetilde{X}$). Passing to the limit as $h\to 0$ we establish existence of the derivative $w_{x_j}\in X(Q_T)$. Continuing these arguments we prove existence of the derivatives $\partial_{x'}^{\alpha'}w\in X(Q_T)$ for any multitindex α'.

By analogy to (30) let

$$v(t,x)\equiv v_0(x)+\int\limits_0^t w(\tau,x)\,d\tau.$$

This function is a solution to the considered problem and $\partial_{x'}^{\alpha'}v\in H^1(Q_T)$ for any multi-tindex α'. Writing down the corresponding analogue of (31) by the same arguments we derive that $\partial_{x'}^{\alpha'}v\in L_2(0,T;H^3(\Omega))$ and, in particular, obtain the solution to the considered problem in the class $L_2(0,T;H^3(\Omega))\cap H^1(Q_T)$. For this solution the inequality (25) is valid and applying closure we derive this inequality in the general case.

Finally note, that for smooth solutions the inequality (26) for $\rho\equiv 1$ succeeds from (14) and for solutions from $\widetilde{X}(Q_T)$ can be obtained via closure. $\square$

Next we establish well-posedness for an auxiliary nonlinear problem with nonlinearity restricted to at most linear rate of growth.

Lemma 4. *Let $(v_0,0,0,u_3,F)\in M_T$ for some $T>0$, $\psi\in L_2(Q_T)$ and the conditions* (11), (16) *be satisfied. Assume in addition, that $b_j=0$ in the inequality* (16) *for all j. Then the problem* (19)–(21) *has a unique solution $v(t,x)$ in the space $\widetilde X(Q_T)$.*

Proof. The proof is based on the contraction principle. For $t_0\in(0,T]$ define on the space $\widetilde X(Q_{t_0})$ a map Λ in such a way: $v=\Lambda u\in\widetilde X(Q_{t_0})$ is a solution in Q_{t_0} to the initial-boundary value problem for the equation

$$v_t-P(\partial_x)v=F-\operatorname{div}_x g(u+\psi)$$

with initial and boundary conditions (20), (21).

Note that

$$\begin{aligned}\|(g_j(u+\psi))_{x_j}\|_{L_2(0,t_0;H^{-1}(\Omega))}&\le\|g_j(u+\psi)\|_{L_2(Q_{t_0})}\\&\le c\|u\|_{L_2(Q_{t_0})}+c\|\psi\|_{L_2(Q_{t_0})}<\infty,\end{aligned}\tag{32}$$

so according to Lemma 3 such a map exists. Moreover, for $u,\widetilde u\in\widetilde X(Q_{t_0})$

$$\begin{aligned}\|(g_j(u+\psi)-g_j(\widetilde u+\psi))_{x_j}\|_{L_2(0,t_0;H^{-1}(\Omega))}&\le c\|u-\widetilde u\|_{L_2(Q_{t_0})}\\&\le ct_0^{1/2}\|u-\widetilde u\|_{C([0,t_0];L_2(\Omega))},\end{aligned}$$

so by virtue of (25) for small t_0, where the value of t_0 does not depend on v_0, u_3, F, ψ, the map Λ is a contraction. □

In the next lemma we establish the main a priori estimate.

Lemma 5. *Let $(v_0,0,0,u_3,F)\in M_T$ for some $T>0$, $\|\psi(t,\cdot)\|_{W^1_\infty(\Omega)}\in L_1(0,T)$, $\psi\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^1(\Omega))$ and the conditions* (11), (16) *be satisfied. Assume in addition, that $b_n<4/n$ in the inequality* (16). *Then there exists a constant $c>0$ uniform with respect to all vector-functions $g(u)$, satisfying the hypothesis of the present lemma with the same constants b_j and $\widetilde c$ in the inequality* (16)*, such that for any solution $v(t,x)$ to the problem* (19)–(21) *in the class $\widetilde X(Q_T)$, $(g_j(v+\psi))_{x_j}\in L_2(0,T;H^{-1}(\Omega))$ for all j, the following estimate is valid:*

$$\|v\|_{\widetilde X(Q_T)}\le c.\tag{33}$$

Proof. Write down the inequality (26), where F is substituted by $F - \operatorname{div}_x g(v+\psi)$. Note that

$$g_j(v+\psi)v_{x_j} = \partial_{x_j} \int\limits_0^v g_j(\theta+\psi)\,d\theta - \psi_{x_j} \int\limits_0^v g_j'(\theta+\psi)\,d\theta. \tag{34}$$

First let $\rho \equiv 1$, then since $b_j \le 1$ in (16) and $v|_{\partial\Omega} = 0$ the equality (34) yields:

$$\left| \int\limits_\Omega g_j(v+\psi)v_{x_j}\,dx \right| \le c\|\psi_{x_j}\|_{L_\infty(\Omega)} \int\limits_\Omega (v^2+\psi^2)\,dx$$

and it follows from (26) that

$$\|v\|_{C([0,T];L_2(\Omega))} \le c. \tag{35}$$

Next let $\rho \equiv 1 + x_n$, then (34) yields:

$$\left| \int\limits_\Omega g_n(v+\psi)\big(v\rho(x_n)\big)_{x_n}\,dx \right| \le c \int\limits_\Omega \Big(|v|^{b_n+2} + |\psi|^{b_n+2}\Big)\,dx \\ + c\|\psi_{x_n}\|_{L_\infty(\Omega)} \int\limits_\Omega (v^2+\psi^2)\,dx.$$

Making use of the interpolational inequality (17), where $p = b_n + 2$, and the already obtained estimate (35) we derive since $nb_n < 4$ that

$$\int\limits_\Omega |v|^{b_n+2}\,dx \le c\bigg(\int\limits_\Omega |\operatorname{grad}_x v|^2\,dx \bigg)^{nb_n/4} \bigg(\int\limits_\Omega v^2\,dx \bigg)^{1-(n-2)b_n/4} \\ + c\bigg(\int\limits_\Omega v^2\,dx \bigg)^{b_n/2+1} \le \varepsilon \int\limits_\Omega |\operatorname{grad}_x v|^2\,dx + c(\varepsilon),$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Therefore, (33) succeeds from (26). $\square$

Now we establish the main existence result.

Theorem 1. *Let* $(u_0, u_1, u_2, u_3, f) \in M_T$ *for some* $T > 0$ *and, in addition,* $\|u_1(t,\cdot)\|_{W^1_\infty(\mathbb{R}^{n-1})}, \|u_2(t,\cdot)\|_{W^1_\infty(\mathbb{R}^{n-1})} \in L_1(0,T)$. *Let the condition* (11) *be satisfied and the functions* $g_j \in C^1(\mathbb{R})$, $j = 1, \ldots, n$, *satisfy the inequality* (16) *for* $b_j \in [0,1]$ *and, in addition,* $b_n < 4/n$. *Then there exists a weak solution* $u(t,x)$ *to the problem* (1)–(3) *from the space* $X(Q_T)$.

Proof. Consider the functions ψ, v_0 and F defined by the formulae (22), (23). Then for these functions the hypothesis of Lemma 5 is satisfied.

For any $h \in (0,1]$ define

$$g_{jh}(u) \equiv \int\limits_0^u \big[g_j'(\theta)\eta(2 - h|\theta|) + g_j'(2h^{-1}\operatorname{sign}\theta)\eta(h|\theta| - 1)\big]\, d\theta.$$

Then for these functions the hypothesis of Lemma 5 is satisfied uniformly with respect to h. On the other hand, $g_{jh}(u) = g_j(u)$ if $|u| \le 1/h$ and $g_{jh}'(u) \le c/h \; \forall\, u$. In particular, the hypothesis of Lemma 4 is satisfied, so for any h there exists a solution $v_h \in \widetilde{X}(Q_T)$ to the corresponding problem (19)–(21), where g_j are substituted by g_{jh}. The estimates (32) and (33) provide that uniformly with respect to h

$$\|u_h\|_{\widetilde{X}(Q_T)} \le c, \tag{36}$$

where u_h, given by the corresponding analogues of the formula (24), are solutions to the problems of the (1)–(3) type, where g_j are substituted by g_{jh}.

Let $l_0 = [n/2] + 2$. Then by virtue of the well-known embedding $L_1(\Omega) \subset H^{1-l_0}(\Omega)$ it follows from the corresponding analogue of the equation (1) itself, that for $0 \le t_1 < t_2 \le T$

$$\|u_h(t_2,\cdot) - u_h(t_1,\cdot)\|_{H^{-l_0}(\Omega)} \le \gamma(t_2 - t_1), \tag{37}$$

where $\gamma(t) \to 0$ as $t \to +0$ and γ does not depend on h.

The estimates (36), (37) by standard arguments (see, e.g., [6]) provide an opportunity to construct the desired solution as a limit of a certain subsequence $\{u_{h'}\}$. $\square$

The next result is concerned with uniqueness and continuous dependence.

Theorem 2. *Let $(u_0, u_1, u_2, u_3, f) \in M_T$ for some $T > 0$, the condition* (11) *be satisfied and the functions $g_j \in C^1(\mathbb{R})$, $j = 1, \dots, n$, satisfy the inequality* (16) *for $b_j \in [0, 2/n]$. Then a weak solution to the problem* (1)–(3) *is unique in the class $X(Q_T)$. Moreover, if for two elements (u_0, u_1, u_2, u_3, f) and $(\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{f})$ of the space M_T there exist solutions $u(t,x)$ and $\widetilde{u}(t,x)$ to the corresponding problems* (1)–(3) *in the space $X(Q_T)$, then*

$$\|u - \widetilde{u}\|_{X(Q_T)} \le c\|(u_0 - \widetilde{u}_0, u_1 - \widetilde{u}_1, u_2 - \widetilde{u}_2, u_3 - \widetilde{u}_3, f - \widetilde{f})\|_{M_T}, \tag{38}$$

where the constant c depends on the norms of u and $\widetilde{u}$ in $X(Q_T)$.

Proof. First of all we establish one auxiliary inequality. Let φ_1, φ_2 be arbitrary functions from $H^1(\Omega)$, then for any $\varepsilon > 0$

$$\left|\int_\Omega |\varphi_1|^{4/n}\varphi_2^2\,dx\right| \le \varepsilon\|\varphi_2\|^2_{H^1(\Omega)} + c(\varepsilon)\|\varphi_1\|^2_{H^1(\Omega)}\|\varphi_1\|^2_{L_2(\Omega)}\|\varphi_2\|^2_{L_2(\Omega)}. \tag{39}$$

In fact, (39) is obtained via application of the Hölder inequality and the inequality (17) for $p_1 = 2n/(n-1)$ and $p_2 = 2n^2/(n^2-2n+2)$:

$$\begin{aligned}\left|\int_\Omega |\varphi_1|^{4/n}\varphi_2^2\,dx\right| &\le \|\varphi_1\|^{4/n}_{L_{p_1}(\Omega)}\|\varphi_2\|^2_{L_{p_2}(\Omega)}\\ &\le c\Big[\|\operatorname{grad}_x\varphi_1\|^{2/n}_{L_2(\Omega)}\|\varphi_1\|^{2/n}_{L_2(\Omega)} + \|\varphi_1\|^{4/n}_{L_2(\Omega)}\Big]\\ &\qquad\times\Big[\|\operatorname{grad}_x\varphi_2\|^{2-2/n}_{L_2(\Omega)}\|\varphi_2\|^{2/n}_{L_2(\Omega)} + \|\varphi_1\|^2_{L_2(\Omega)}\Big].\end{aligned}$$

Let ψ, v_0, F be defined by the formulae (22), (23) and $\widetilde{\psi}$, $\widetilde{v}_0$, $\widetilde{F}$ – by the similar ones, where u_0, u_1, u_2, f are substituted by $\widetilde{u}_0$, $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{f}$. Then the function $v \equiv (u-\widetilde{u}) - (\psi - \widetilde{\psi})$ is a solution in $X(Q_T)$ to the linear problem

$$\begin{gathered} v_t - P(\partial_x)v = (F-\widetilde{F}) - (\operatorname{div}_x g(u) - \operatorname{div}_x g(\widetilde{u})), \quad (t,x)\in Q_T,\\ v\big|_{t=0} = v_0 - \widetilde{v}_0, \quad x\in\Omega,\\ v\big|_{x_n=0} = v\big|_{x_n=1} = 0, \quad v_{x_n}\big|_{x_n=1} = u_3 - \widetilde{u}_3.\end{gathered}$$

Applying (39), where either $\varphi_1 = \varphi_2 \equiv u$ or $\varphi_1 = \varphi_2 \equiv \widetilde{u}$, we derive that $\operatorname{div}_x g(u), \operatorname{div}_x g(\widetilde{u}) \in L_2(0,T;H^{-1}(\Omega))$.

Write down for the function $v(t,x)$ the corresponding inequality (26) in the case $\rho \equiv 1 + x_n$. Note that

$$\big|g_j(u) - g_j(\widetilde{u})\big| \le c\big(|v| + |\psi - \widetilde{\psi}|\big)\big(|u|^{2/n} + |\widetilde{u}|^{2/n} + 1\big).$$

Again using the inequality (39) it is easy to see that for any $\varepsilon > 0$

$$\begin{aligned}\left|\int_\Omega \big(g_j(u) - g_j(\widetilde{u})\big)v_{x_j}\,dx\right| &\le \varepsilon\int_\Omega |\operatorname{grad}_x v|^2\,dx + c\|\psi-\widetilde{\psi}\|^2_{H^1(\Omega)}\\ &+ c(\varepsilon)\left(\|u\|^2_{H^1(\Omega)}\|u\|^2_{L_2(\Omega)} + \|\widetilde{u}\|^2_{H^1(\Omega)}\|\widetilde{u}\|^2_{L_2(\Omega)} + 1\right)\\ &\qquad\times\int_\Omega \big(v^2 + (\psi-\widetilde{\psi})^2\big)\,dx,\end{aligned}$$

whence (38) follows. $\square$

Remark 2. Under the hypotheses both of Theorem 1 and Theorem 2 the problem (1)–(3) is globally well-posed in the corresponding spaces.

Remark 3. The assumptions on the functions u_1 and u_2 seem not to be optimal. In the paper [6] a special solution of the "boundary potential" type to the problem in Π_T^+ for the corresponding linearized equation was constructed, which provided relaxation of the smoothness assumptions on u_1 in comparison with the present paper. A similar approach may be useful for the present problem also.

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