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More Examples of Hereditarily ℓ_p Banach Spaces

Mikhail M. Popov

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Abstract. Extending our previous result, we construct a class of hereditarily ℓ_p for $1 \le p < \infty$ (c_0) Banach spaces, investigate their properties, and show that the classical Pitt theorem on compactness of operators from ℓ_s to ℓ_p for $1 \le p < s < \infty$ is false in the general setting of hereditarily ℓ_s and ℓ_p spaces.

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1. Preliminaries and Introduction

We use the standard terminology and usual notations as in [5-7]. By $[x_i]_{i=1}^{\infty}$ we denote the closed linear span of a sequence $\{x_i\}_{i=1}^{\infty}$ in a Banach space X. $S(X)$ stands for the unit sphere of a Banach space X. By a "subspace" of a Banach space we mean a closed linear subspace.

Recall that an infinite dimensional Banach space X is said to be hereditarily Y (Y is a Banach space), if each infinite dimensional subspace X_0 of X contains a further subspace $Y_0 \subseteq X_0$ which is isomorphic to Y. Thus, if X is hereditarily Y then we naturally expect to have the interior properties of X to be close to that of Y. Any exception may be of interest. So, it is well known that ℓ_1 possesses the Schur property (a Banach space X is said to have the Schur property provided weak convergence of sequences in X implies their norm convergence), while there are hereditarily ℓ_1 Banach spaces without the Schur property [3], [2], [9]. A hereditarily ℓ_2 Banach space need not be reflexive, a counterexample is the James tree JT [4]. See also a recent paper of P. Azimi [1], where the author makes an attempt to generalize the idea of [2] for constructing new hereditarily ℓ_p Banach spaces.

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Using the main idea of [9], we construct classes of hereditarily ℓ_p, $1 \leq p < \infty$ (and respectively, c_0) Banach sequence spaces Z_p. Section 3 is devoted to a proof that ℓ_p (resp., c_0) is isomorphic to a complemented subspace of Z_p, which is used below. Section 4 is devoted to a study of the question whether the classical Pitt theorem (that if $1 \leq p < s < \infty$ then every continuous linear operator from ℓ_s to ℓ_p is compact) remains true for the setting of hereditarily ℓ_s and ℓ_p spaces instead of the ℓ_s and ℓ_p themselves. We show that in general, the answer is negative, but nevertheless not everything is clear in this emphasis. We state some open questions in the last section and do some historical comments on them. We note also that for some values of parameters our spaces Z_p are isometrically embedded in the classical function spaces $L_p = L_p[0,1]$.

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We recall that for an arbitrary sequence of Banach spaces $\{X_n\}_{n=1}^\infty$ and any number $p \in [1, \infty)$ the direct sum of these spaces in the sense of ℓ_p is defined as the linear space

$$X = \left(\sum_{n=1}^\infty \oplus X_n \right)_p$$

of all sequences $x = (x_1, x_2, \cdots)$, $x_n \in X_n$, $n = 1, 2, \cdots$ for which

$$\|x\| = \left(\sum_{n=1}^\infty \|x_n\|^p \right)^{\frac{1}{p}} < \infty,$$

where the norm $\|x_n\|$ is considered in the corresponding space X_n. Analogously, the direct sum of the spaces $\{X_n\}_{n=1}^\infty$ in the sense of c_0 is defined as the linear space

$$X = \left(\sum_{n=1}^\infty \oplus X_n \right)_0$$

of all sequences $x = (x_1, x_2, \cdots)$, $x_n \in X_n$, $n = 1, 2, \cdots$ for which $\lim_n \|x_n\| = 0$ with the norm

$$\|x\| = \max_n \|x_n\|.$$

Fix any decreasing sequence $\mathcal{P}$ of reals $p_1 > p_2 > \cdots > 1$ (note that we do not care if p_n tends to 1 or not). Consider any fixed value of p from the set $p \in \{0\} \cup [1, \infty)$ and the following corresponding sequence space

$$X_p^{\mathcal{P}} = \left(\sum_{n=1}^\infty \oplus \ell_{p_n} \right)_p,$$

where the direct sum is considered in the sense of ℓ_p, $p \geq 1$ or c_0 if $p = 0$.

For each $n \geq 1$, denote by $\{\overline{e}_{i,n}\}_{i=1}^{\infty}$ the unit vector basis of ℓ_{p_n} and by $\{e_{i,n}\}_{i=1}^{\infty}$ its natural copy in $X_p^{\mathcal{P}}$:

$$e_{i,n} = \Big(\underbrace{0, \cdots, 0}_{n-1}, \overline{e}_{i,n}, 0, \cdots\Big) \in X_p^{\mathcal{P}}.$$

Let $\delta_n > 0$ be such reals that for $\Delta = (\delta_1, \delta_2, \cdots)$ we have $\|\Delta\|_p = 1$ (i.e. $\sum_{n=1}^{\infty} \delta_n^p = 1$ if $p \geq 1$, and $\lim_n \delta_n = 0$ and $\max_n \delta_n = 1$ if $p = 0$.

For $i \geq 1$ put $z_i = \sum_{n=1}^{\infty} \delta_n e_{i,n}$. Evidently, $\|z_i\| = 1$ for each i.

Denote by $Z_p = Z_p(\mathcal{P})$ the closed linear span of $\{z_i\}_{i=1}^{\infty}$ (formally, Z_p depends also on Δ, but actually nothing would change if we replace one value of Δ by another and hence we fix Δ from now on). We show that Z_p is hereditarily ℓ_p if $p \geq 1$ and c_0 if $p = 0$. Note that this construction is a generalized version of [9] and that this fact is actually proved for $p = 1$ in [9].

There is an essential difference between the cases $p = 0, 1$ and $1 < p < \infty$. For $X = \ell_1$ or $X = c_0$, every Banach space isomorphic to X for arbitrary $\varepsilon > 0$ contains a subspace which is $(1+\varepsilon)$-isomorphic to X [6,p.97], while this is false for $X = \ell_p$ when $1 < p < \infty$ [8,p.1348] (recall that Banach spaces X and Y are said to be λ-isomorphic provided there exists an isomorphism $T : X \to Y$ with $\|T\| \cdot \|T^{-1}\| \leq \lambda$; evidently, $\lambda \geq 1$ in this case). Thus, when speaking of hereditarily ℓ_1 or c_0 spaces, it is enough to say "subspace isomorphic to X" and by "X is hereditarily ℓ_p" we mean the strongest $(1+\varepsilon)$-isomorphic version, i.e. each infinite dimensional subspace X_0 of X for every $\varepsilon > 0$ contains a further subspace $Y_0 \subseteq X_0$ which is $(1+\varepsilon)$-isomorphic to ℓ_p.

Now we recall some notions on bases in Banach spaces. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *basis* for X if for each $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$ in the sense of the norm convergence in X. By a theorem of S. Banach [6,p.1], the following so-called *projections associated with the basis* $\{x_n\}_{n=1}^{\infty}$

$$P_n\Big(\sum_{k=1}^{\infty} a_k x_k\Big) = \sum_{k=1}^{n} a_k x_k,$$

are uniformly bounded, and the number $K = \sup_n \|P_n\|$ is called the *basis constant* of the basis $\{x_n\}_{n=1}^{\infty}$. A sequence which is a basis for its closed linear span is called a *basic sequence.* A *block basis* of a basic sequence

$\{x_n\}_{n=1}^\infty$ is any sequence of non-zero elements of the form

$$u_k = \sum_{j=n_k+1}^{n_{k+1}} a_j x_j, \ \ k = 1, 2, \cdots,$$

where $0 = n_1 < n_2 < \cdots$ — some increasing sequence of integers. Evidently, a block basis is also a basic sequence whose basis constant is less or equal to that of the basic sequence. Two basic sequences $\{x_n\}_{n=1}^\infty$ in X and $\{y_n\}_{n=1}^\infty$ in Y are said to be λ-equivalent if there exists an isomorphism $T : [x_i]_{i=1}^\infty \to [y_i]_{i=1}^\infty$ with $\|T\| \cdot \|T^{-1}\| \leq \lambda$. Basic sequences are called equivalent if they are λ-equivalent for some $\lambda \geq 1$. A basis $\{x_n\}_{n=1}^\infty$ in a Banach space X is said to be *symmetric* if for any permutation π of integers the sequence $\{x_{\pi(n)}\}_{n=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$. If they are 1-equivalent for any permutation π then the basis is called 1-symmetric.

2. The Proof that Z_p is Hereditarily ℓ_p

For each $I \subseteq \mathbb{N}$ by P_I we denote the natural projection of $X_p^{\mathcal{P}}$ onto $[e_{i,n} : \ i \in \mathbb{N}, \ n \in I]$ (i.e. with the kernel $[e_{i,n} : \ i \in \mathbb{N}, \ n \notin I]$). Of course, $\|P_I\| = \|Id - P_I\| = 1$. Given an infinite dimensional subspace E_0 of Z_p, we find a sequence $\{x_s\}_{s=1}^\infty$ in E_0 and a block basic subsequence $\{u_s\}_{s=1}^\infty$ of $\{z_i\}_{i=1}^\infty$ having "almost disjoint supports" and which is close enough to $\{x_s\}_{s=1}^\infty$. (Here by "almost disjoint supports" we mean that for each $\varepsilon > 0$ there are disjoint subsets I_s of $\mathbb{N}$ with $\|P_{I_s} u_s\| \geq (1-\varepsilon)\|u_s\|$). Hence $\{x_s\}_{s=1}^\infty$ contains a subsequence equivalent to the unit vector basis of ℓ_p.

Lemma 2.1. *For all scalars $\{a_i\}_{i=1}^m$ and each permutation of integers $\tau : \mathbb{N} \to \mathbb{N}$ one has*

$$\left\| \sum_{i=1}^m a_i z_{\tau(i)} \right\|^p = \sum_{n=1}^\infty \delta_n^p \left(\sum_{i=1}^m |a_i|^{p_n} \right)^{\frac{p}{p_n}}, \quad \textit{if } 1 \leq p < \infty$$

and

$$\left\| \sum_{i=1}^m a_i z_{\tau(i)} \right\| = \sup_{n \in \mathbb{N}} \ \delta_n \left(\sum_{i=1}^m |a_i|^{p_n} \right)^{\frac{1}{p_n}}, \quad \textit{if } p = 0.$$

Hence, $\{z_i\}_{i=1}^\infty$ is a 1-symmetric basic sequence.

Proof. The proof is straightforward:

$$\left\| \sum_{i=1}^m a_i z_{\tau(i)} \right\|^p = \sum_{n=1}^\infty \delta_n^p \left\| \sum_{i=1}^m a_i e_{\tau(i),n} \right\|^p = \sum_{n=1}^\infty \delta_n^p \left(\sum_{i=1}^m |a_i|^{p_n} \right)^{\frac{p}{p_n}}$$

for $1 \le p < \infty$ and

$$\left\| \sum_{i=1}^{m} a_i z_{\tau(i)} \right\| = \sup_{n\in\mathbb{N}} \delta_n \left\| \sum_{i=1}^{m} a_i e_{\tau(i),n} \right\| = \sup_{n\in\mathbb{N}} \delta_n \left(\sum_{i=1}^{m} |a_i|^{p_n} \right)^{\frac{1}{p_n}}$$

for $p = 0$. ☐

Thus, if a series $\sum\limits_{i=1}^{\infty} a_i z_i$ converges then $\sum\limits_{i=1}^{\infty} |a_i|^{p_n} < \infty$ for each n and $\lim\limits_{n} \delta_n \left(\sum\limits_{i=1}^{m} |a_i|^{p_n} \right)^{\frac{1}{p_n}} = 0$.

The following lemma as well as its proof exactly coincides with the corresponding lemma from [9]. To make our note self-contained, we provide it with a complete proof.

Lemma 2.2. *Let E_0 be an infinite dimensional subspace of Z_p, $n, m, j \in \mathbb{N}$ $(n > 1)$ and $\varepsilon > 0$. Then there are $\{x_i\}_{i=1}^m \subset E_0$ and $\{u_i\}_{i=1}^m \subset Z_p$ of the form*

$$u_i = \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} z_s \;\; \text{where } j = j_1 < j_2 < \ldots < j_{m+1}$$

such that

$$\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} = 1 \;\; \text{and } \|u_i - x_i\| < \frac{\varepsilon}{m} \|u_i\|$$

for each $i = 1, \ldots, m$.

Proof. Put $E_1 = E_0 \bigcap [z_i]_{i=j+1}^{\infty}$. Since E_0 is infinite dimensional and $[z_i]_{i=j+1}^{\infty}$ has finite codimension in Z_p, E_1 is infinite dimensional as well. Put $j_1 = j$ and choose any

$$\overline{x}_1 = \sum_{s=j_1+1}^{\infty} \overline{a}_{1,s} z_s \in E_1 \setminus \{0\}.$$

Without lost of generality we may assume that

$$\sum_{s=j_1+1}^{\infty} |\overline{a}_{1,s}|^{p_{n-1}} = 1$$

(otherwise we multiply $\overline{x}_1$ by a suitable number). Then choose $j_2 > j_1$ so that for

$$\overline{u}_1 = \sum_{s=j_1+1}^{j_2} \overline{a}_{1,s} z_s$$

we have

$$\|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \|\overline{x}_1\|}{4m}, \quad \lambda_1 = \Big(\sum_{s=j_1+1}^{j_2} |\overline{a}_{1,s}|^{p_{n-1}} \Big)^{\frac{1}{p_{n-1}}} \geq \frac{1}{2}$$

and

$$\|\overline{u}_1\| \geq \frac{\|\overline{x}_1\|}{2}.$$

Hence,

$$\|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \|\overline{u}_1\|}{2m}$$

Now put $a_{1,s} = \lambda_1^{-1} \overline{a}_{1,s}$, $x_1 = \lambda_1^{-1} \overline{x}_1$ and $u_1 = \lambda_1^{-1} \overline{u}_1$. Then

$$\sum_{s=j_1+1}^{j_2} |a_{1,s}|^{p_{n-1}} = \frac{1}{\lambda_1^{p_{n-1}}} \sum_{s=j_1+1}^{j_2} |\overline{a}_{1,s}|^{p_{n-1}} = 1$$

and

$$\|u_1 - x_1\| = \frac{1}{\lambda_1} \|\overline{u}_1 - \overline{x}_1\| < \frac{\varepsilon \, \|\overline{u}_1\|}{2\lambda_1 \, m} \leq \frac{\varepsilon \, \|\overline{u}_1\|}{m} \leq \frac{\varepsilon \, \|u_1\|}{m}.$$

Continuing the procedure in the obvious manner, we construct the desired sequences. □

For $n \in \mathbb{N}$ denote $Q_n = P_{\{n, n+1, \cdots\}}$.

Lemma 2.3. *Let E_0 be an infinite dimensional subspace of Z_p, $j, n \in \mathbb{N}$ and $\varepsilon > 0$. There exist an $x \in E_0$, $x \neq 0$ and a $u \in Z_p$ of the form*

$$u = \sum_{i=j+1}^{l} a_i z_i, \quad \textit{where } l > j$$

such that

(i) $\|Q_n u\| \geq (1 - \varepsilon) \, \|u\|$;

(ii) $\|x - u\| < \varepsilon \, \|u\|$.

Proof. Choose $m \in \mathbb{N}$ so that

$$\frac{1}{\delta_n^p} \, m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} < \varepsilon \quad \text{or} \quad \frac{1}{\delta_n} \, m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} < \varepsilon \quad \text{if } p = 0.$$

Using Lemma 2.2, choose $\{x_i\}_{i=1}^m \subset E_0$ and $\{u_i\}_{i=1}^m \subset Z_p$ to satisfy the claims of the lemma and put

$$x = \sum_{i=1}^{m} x_i \quad \text{and} \quad u = \sum_{i=1}^{m} u_i.$$

First, we prove (ii). Since $\{z_s\}_{s=1}^{\infty}$ is 1-symmetric then $\|u_i\| \leq \|u\|$ for $i = 1, ..., m$ and

$$\|x - u\| \leq \sum_{i=1}^{m} \|x_i - u_i\| < \sum_{i=1}^{m} \frac{\varepsilon \|u_i\|}{m} \leq \sum_{i=1}^{m} \frac{\varepsilon \|u\|}{m} = \varepsilon \|u\|.$$

To prove (i), we first show that

$$\|u\| - \|Q_n u\| < m^{\frac{1}{p_{n-1}}}.$$

Anyway, $\|u\| - \|Q_n u\| \leq \|P_{\{1,\cdots,n-1\}} u\|$. Hence, for $p \geq 1$ one has

$$\begin{aligned}
\Big(\|u\| - \|Q_n u\|\Big)^p &\leq \sum_{k=1}^{n-1} \delta_k^p \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k} \right\|^p \\
&= \sum_{k=1}^{n-1} \delta_k^p \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_k} \right)^{\frac{p}{p_k}} \leq \sum_{k=1}^{n-1} \delta_k^p \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \right)^{\frac{p}{p_{n-1}}} \\
&= \sum_{k=1}^{n-1} \delta_k^p \left(\sum_{i=1}^{m} 1 \right)^{\frac{p}{p_{n-1}}} = m^{\frac{p}{p_{n-1}}} \sum_{k=1}^{n-1} \delta_k^p < m^{\frac{p}{p_{n-1}}}
\end{aligned}$$

and for $p = 0$

$$\begin{aligned}
\|u\| - \|Q_n u\| &\leq \max_{1 \leq k < n} \delta_k \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k} \right\| \\
&= \max_{1 \leq k < n} \delta_k \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_k} \right)^{\frac{1}{p_k}} \leq \max_{1 \leq k < n} \delta_k \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \right)^{\frac{1}{p_{n-1}}} \\
&= \max_{1 \leq k < n} \delta_k \left(\sum_{i=1}^{m} 1 \right)^{\frac{1}{p_{n-1}}} = m^{\frac{1}{p_{n-1}}} \max_{1 \leq k < n} \delta_k \leq m^{\frac{1}{p_{n-1}}}.
\end{aligned}$$

On the other hand, for $p \geq 1$

$$\begin{aligned}
\|u\|^p &= \sum_{k=1}^{\infty} \delta_k^p \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k} \right\|^p \geq \delta_n^p \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,n} \right\|^p \\
&= \delta_n^p \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n} \right)^{\frac{p}{p_n}} \geq \delta_n^p \left(\sum_{i=1}^{m} \left(\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \right)^{\frac{p_n}{p_{n-1}}} \right)^{\frac{p}{p_n}} \\
&= \delta_n^p \left(\sum_{i=1}^{m} 1 \right)^{\frac{p}{p_n}} = \delta_n^p \, m^{\frac{p}{p_n}}
\end{aligned}$$

and for $p = 0$

$$\|u\| = \max_{k\in\mathbb{N}} \delta_k \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,k} \right\| \geq \delta_n \left\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} e_{s,n} \right\|$$
$$= \delta_n \left(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n} \right)^{\frac{1}{p_n}} \geq \delta_n \left(\sum_{i=1}^{m} \left(\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \right)^{\frac{p_n}{p_{n-1}}} \right)^{\frac{1}{p_n}}$$
$$= \delta_n \left(\sum_{i=1}^{m} 1 \right)^{\frac{1}{p_n}} = \delta_n m^{\frac{1}{p_n}}.$$

Thus, anyway $\|u\| \geq \delta_n m^{\frac{1}{p_n}}$ and hence

$$1 - \frac{\|Q_n u\|}{\|u\|} \leq \frac{1}{\delta_n} \, m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} < \varepsilon$$

and $\|Q_n u\| \geq (1-\varepsilon)\, \|u\|$. □

Lemma 2.4. *Suppose $\varepsilon > 0$ and ε_s for $s \in \mathbb{N}$ are such that:*

$2\varepsilon_s \leq \varepsilon$ *if* $p = 1$;

$\sum\limits_{s=1}^{\infty} (2\varepsilon_s)^q \leq \varepsilon^q$ *if* $1 < p < \infty$ *where* $\frac{1}{p} + \frac{1}{q} = 1$;

$\sum\limits_{s=1}^{\infty} 2\varepsilon_s \leq \varepsilon$ *if* $p = 0$.

If for given vectors $\{u_s\}_{s=1}^{\infty} \subset S(Z_p)$ where $Z_p = Z_p(\mathcal{P})$, there is a sequence of integers $1 \leq n_1 < n_2 < \cdots$ such that the following two conditions hold

(i) $$\|u_s - Q_{n_s} u_s\| \leq \varepsilon_s,$$

(ii) $$\|Q_{n_{s+1}} u_s\| \leq \varepsilon_s$$

for each $s \in \mathbb{N}$ then $\{u_s\}_{s=1}^{\infty}$ is $(1+\varepsilon)(1-3\varepsilon)^{-1}$-equivalent to the unit vector basis of ℓ_p (respectively, c_0).

Proof. Put $v_s = Q_{n_s} u_s - Q_{n_{s+1}} u_s$ for $s \in \mathbb{N}$. Since $v_s = u_s - (u_s - Q_{n_s} u_s + Q_{n_{s+1}} u_s)$, then $\|v_s\| \geq 1 - 2\varepsilon_s > 1 - 2\varepsilon$. On the other hand, by definitions of Q_i and the norm on Z_p one has $\|v_s\| \leq \|u_s\| = 1$. Thus, $1 - 2\varepsilon < \|v_s\| \leq 1$ for each $s \in \mathbb{N}$. Then for each scalars $\{a_s\}_{s=1}^{m}$ one has

$$(1-2\varepsilon)^p \sum_{s=1}^{m} |a_s|^p \leq \sum_{s=1}^{m} |a_s|^p \|v_s\|^p = \left\| \sum_{s=1}^{m} a_s v_s \right\|^p \leq \sum_{s=1}^{m} |a_s|^p \qquad (1)$$

for $1 \le p < \infty$ and

$$(1-2\varepsilon)\max_{1\le s\le m}|a_s| \le \max_{1\le s\le m}|a_s|\|v_s\| = \left\|\sum_{s=1}^m a_s v_s\right\| \le \max_{1\le s\le m}|a_s| \qquad (2)$$

for $p = 0$. By the lemma conditions

$$\left\|\sum_{s=1}^m a_s(u_s - v_s)\right\| \le \left\|\sum_{s=1}^m a_s(u_s - Q_{n_s}u_s)\right\| + \left\|\sum_{s=1}^m a_s Q_{n_{s+1}}u_s\right\|$$
$$\le \sum_{s=1}^m |a_s|\|u_s - Q_{n_s}u_s\| + \sum_{s=1}^m |a_s|\|Q_{n_{s+1}}u_s\| \le \sum_{s=1}^m |a_s|\, 2\varepsilon_s$$

then depending on p:

$$\le \left(\sum_{s=1}^m |a_s|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{s=1}^m (2\varepsilon_s)^q\right)^{\frac{1}{q}} < \varepsilon \left(\sum_{s=1}^m |a_s|^p\right)^{\frac{1}{p}} \qquad (3)$$

if $1 < p < \infty$,

$$\le \sum_{s=1}^m |a_s|^p \cdot \max_{1\le s\le m} 2\varepsilon_s \le \varepsilon \sum_{s=1}^m |a_s| \qquad (4)$$

if $p = 1$ and

$$\le \max_{1\le s\le m}|a_s| \cdot \sum_{s=1}^m 2\varepsilon_s < \varepsilon \max_{1\le s\le m}|a_s| \qquad (5)$$

if $p = 0$. Using $(1) - (5)$ we obtain

$$\left\|\sum_{s=1}^m a_s u_s\right\| \ge \left\|\sum_{s=1}^m a_s v_s\right\| - \left\|\sum_{s=1}^m a_s(u_s - v_s)\right\|$$

depending on p:

$$\ge (1-2\varepsilon)\left(\sum_{s=1}^m |a_s|^p\right)^{\frac{1}{p}} - \varepsilon\left(\sum_{s=1}^m |a_s|^p\right)^{\frac{1}{p}} = (1-3\varepsilon)\left(\sum_{s=1}^m |a_s|^p\right)^{\frac{1}{p}} \qquad (6)$$

if $1 \le p < \infty$ and

$$\ge (1-2\varepsilon)\max_{1\le s\le m}|a_s| - \varepsilon\max_{1\le s\le m}|a_s| = (1-3\varepsilon)\max_{1\le s\le m}|a_s| \qquad (7)$$

if $p = 0$. On the other hand,

$$\left\|\sum_{s=1}^m a_s u_s\right\| \le \left\|\sum_{s=1}^m a_s v_s\right\| + \left\|\sum_{s=1}^m a_s(u_s - v_s)\right\|$$

depending on p:

$$\leq \Big(\sum_{s=1}^{m}|a_s|^p\Big)^{\frac{1}{p}} + \varepsilon\Big(\sum_{s=1}^{m}|a_s|^p\Big)^{\frac{1}{p}} = (1+\varepsilon)\Big(\sum_{s=1}^{m}|a_s|^p\Big)^{\frac{1}{p}} \tag{8}$$

if $1 \leq p < \infty$ and

$$\leq \max_{1\leq s\leq m}|a_s| + \varepsilon \max_{1\leq s\leq m}|a_s| = (1+\varepsilon)\max_{1\leq s\leq m}|a_s| \tag{9}$$

if $p = 0$. Combining (6)–(9) we obtain the desired inequalities

$$(1-3\varepsilon)\Big(\sum_{s=1}^{m}|a_s|^p\Big)^{\frac{1}{p}} \leq \Big\|\sum_{s=1}^{m}a_s u_s\Big\| \leq (1+\varepsilon)\Big(\sum_{s=1}^{m}|a_s|^p\Big)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$(1-3\varepsilon)\max_{1\leq s\leq m}|a_s| \leq \Big\|\sum_{s=1}^{m}a_s u_s\Big\| \leq (1+\varepsilon)\max_{1\leq s\leq m}|a_s|$$

for $p = 0$. □

Theorem 2.1. *The Banach space $Z_p = Z_p(\mathcal{P})$ is hereditarily ℓ_p if $1 \leq p < \infty$ and is hereditarily c_0 if $p = 0$.*

Proof. Let E_0 be an infinite dimensional subspace of Z_p and fix an $\varepsilon > 0$, quite enough small to satisfy $(1+\varepsilon)(1-3\varepsilon)^{-1} \leq 2$. Choose any sequence of positive numbers ε_s to satisfy the conditions of Lemma 2.4. Then choose by the Krein-Milman-Rutman stability of basic sequences theorem [6,p.5] numbers $\eta_s > 0,\ s \in \mathbb{N}$ such that if $\{x_n\}$ is a basic sequence in a Banach space X with the basis constant $\leq K$ and y_s are vectors in X with $\|x_s - y_s\| < (2K)^{-1}\eta_s$ then $\{y_s\}$ is also a basic sequence which is $(1+\varepsilon)$-equivalent to $\{x_s\}$. Using Lemma 2.3, construct inductively sequences $\{x_s\}_{s=1}^{\infty} \subset E_0,\ \{u_s\}_{s=1}^{\infty} \subset Z_p$ of the form

$$u_s = \sum_{i=j_s+1}^{j_{s+1}} a_i z_i,$$

where $j_1 < j_2 < ...$ and $\|u_s\| = 1$ and a sequence $1 \leq n_1 < n_2 < \cdots$ so that

$$(i) \qquad \|Q_{n_s}u_s\| \geq 1 - \varepsilon_s\,,$$

$$(ii) \qquad \|u_s - x_s\| \leq \frac{\eta_s}{4}\,,$$

(iii) $$\|Q_{n_{s+1}} u_s\| < 1 - \varepsilon_s\,.$$

To see that this can be done, let $j_1 = n_1 = 1$. Choose by Lemma 2.3 an $x_1 \in Z_p \setminus \{0\}$ and

$$u_1 = \sum_{i=j_1+1}^{j_2} a_i z_i$$

such that $\|u_1\| = 1$, $\|Q_{n_1} u_1\| \geq 1 - \varepsilon_1$ and $\|x_1 - u_1\| < 4^{-1}\delta_1$. Then choose $n_2 > n_1$ so that $\|Q_{n_2} u_1\| < \varepsilon_1$. Continuing the procedure in the obvious manner, we construct the desired sequences.

Evidently, (i) yields

(i′) $$\|u_s - Q_{n_s} u_s\| \leq \varepsilon_s.$$

Conditions *(i′)* and *(iii)* imply that $\{u_s\}_{s=1}^{\infty}$ is $(1+\varepsilon)(1-3\varepsilon)^{-1}$-equivalent to the unit vector basis of ℓ_p (respectively, c_0), by Lemma 2.4. Then by the choice of $\{\eta_s\}_{s=1}^{\infty}$, $\{x_s\}_{s=1}^{\infty}$ is a basic sequence $(1+\varepsilon)$-equivalent to $\{u_s\}_{s=1}^{\infty}$. Thus, $\{x_s\}_{s=1}^{\infty}$ is $(1+\varepsilon)^2(1-3\varepsilon)^{-1}$-equivalent to the unit vector basis of ℓ_p (respectively, c_0). □

3. $Z_p(\mathcal{P})$ Contains a Complemented Copy of ℓ_p

Recall that a subspace X of a Banach space Z is called *complemented* if there exists a subspace Y of Z such that Z can be decomposed into a direct sum $Z = X \oplus Y$. Of course, for each subspace X of Z there are a lot of linear subspaces $Y \subseteq Z$ such that $Z = X \oplus Y$, but it may happen that all of them are not closed. In other words, a subspace X of Z is complemented if it is the range of some linear bounded projection of Z onto X.

Theorem 3.1. 1. *The space $Z_p = Z_p(\mathcal{P})$ contains a complemented subspace isomorphic to ℓ_p (resp., c_0) for each p and $\mathcal{P}$.*

2. *The space $Z_p(\mathcal{P}) \oplus \ell_p$ is isomorphic to Z_p (respectively, $Z_0(\mathcal{P}) \oplus c_0$).*

Proof. For $j, m \in \mathbb{N}$ we set $\tilde{u}_{j,m} = z_{j+1} + \cdots + z_{j+m}$ and $u_{j,m} = \|\tilde{u}_{j,m}\|^{-1}\tilde{u}_{j,m}$.

We prove the following statement **(A)**: for each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there is an m_0 such that for every $j \in \mathbb{N}$ and every $m \geq m_0$ we have $\|u_{j,m} - Q_n u_{j,m}\| < \varepsilon$. Indeed, for $1 \leq p < \infty$

$$\|u_{j,m} - Q_n u_{j,m}\|^p = \frac{\|\tilde{u}_{j,m} - Q_n \tilde{u}_{j,m}\|^p}{\|\tilde{u}\|^p} = \frac{\sum\limits_{s=1}^{n-1} \delta_s^p\, m^{\frac{p}{p_s}}}{\sum\limits_{s=1}^{\infty} \delta_s^p\, m^{\frac{p}{p_s}}} \leq \frac{\sum\limits_{s=1}^{n-1} \delta_s^p\, m^{\frac{p}{p_{n-1}}}}{\delta_n^p\, m^{\frac{p}{p_n}}}$$

$$< \frac{m^{\frac{p}{p_{n-1}}}}{\delta_n^p \; m^{\frac{p}{p_n}}} = \delta_n^{-p} \; m^{\frac{p}{p_{n-1}} - \frac{p}{p_n}} \quad \to 0 \quad \text{as } m \to \infty$$

and for $p = 0$

$$\|u_{j,m} - Q_n u_{j,m}\| = \frac{\max\limits_{1 \le s < n} \delta_s \; m^{\frac{1}{p_s}}}{\sup\limits_{1 \le s < \infty} \delta_s \; m^{\frac{1}{p_s}}} \le \frac{m^{\frac{1}{p_{n-1}}}}{\delta_n \; m^{\frac{1}{p_n}}} = \delta_n^{-1} \; m^{\frac{1}{p_{n-1}} - \frac{1}{p_n}} \quad \to 0$$

as $m \to \infty$ and **(A)** is proved.

Then, using an inductive procedure, prove the following fact **(B)**: given a sequence of positive numbers $\{\varepsilon_s\}_{s=1}^{\infty}$, there exist sequences of integers $1 = j_1 < j_2 < \cdots$ and $1 = n_1 < n_2 < \cdots$ such that for

$$\tilde{u}_s = \tilde{u}_{j_s, j_{s+1} - j_s} = z_{j_s + 1} + \cdots + z_{j_{s+1}} \quad \text{and} \quad u_s = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}$$

we have

(*i*) $$\|u_s - Q_{n_s} u_s\| \le \varepsilon_s,$$

(*ii*) $$\|Q_{n_{s+1}} u_s\| \le \varepsilon_s$$

for each $s \in \mathbb{N}$.

Indeed, put $j_1 = n_1 = 1$ and $j_2 = 2$. Then we have $u_1 = z_2$ and $Q_{n_1} u_1 = u_1$ and hence (*i*) is trivially satisfied for $s = 1$. Then choose $n_2 > n_1$ to satisfy (*ii*) for $s = 1$, i.e. so that $\|Q_{n_2} u_1\| < \varepsilon_1$. Then using **(A)**, choose $j_2 > j_1$ to satisfy (*i*). Continuing the procedure in the obvious manner, we construct the desired sequences.

Now applying to **(B)** Lemma 2.4, we obtain the following statement **(C)**: for each $\varepsilon > 0$ there exists a sequence $\{\sigma_j\}_{j=1}^{\infty}$ of disjoint nonempty finite subsets of $\mathbb{N}$ with $\max \sigma_j < \min \sigma_{j+1}$ such that the corresponding block basis with constant coefficients of the basis $\{z_i\}_{i=1}^{\infty}$

$$u_s = \sum_{n \in \sigma_s} z_n$$

spans a subspace E, $(1 + \varepsilon)$-isomorphic to ℓ_p (resp., c_0). By [6,p.116], E is complemented and the claim 1 of the theorem is proved. Claim 2 follows from [6,p.117]. □

4. Operators between $Z_{p_1}(\mathcal{P}_1)$ and $Z_{p_2}(\mathcal{P}_2)$

Definition 4.1. *Let X and Y be any of the spaces $\ell_p(1 \le p < \infty)$, c_0, Z_p $(1 \le p < \infty,\ p = 0)$ with their natural bases $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ respectively. The formal (maybe, unbounded) operator $T : X \to Y$ which extends by linearity and continuity the equality $Tx_n = y_n$ we shall call **the natural operator** from X to Y.*

Proposition 4.1. *Let $p \in \{0\} \cup [1, +\infty)$, $\mathcal{P}$ be arbitrary, as above.*
(i) If $\inf\limits_n p_n < p$ then the natural operator from ℓ_p to Z_p is unbounded.
(ii) If $\inf\limits_n p_n \ge p$ then the natural operator from Z_p to ℓ_p is unbounded.

Proof. For constant scalars $a_1 = a_2 = \cdots = a_m = 1$ we have by Lemma 2.1

$$\Big\|\sum_{i=1}^m z_i\Big\|^p = \sum_{n=1}^\infty \delta_n^p\ m^{\frac{p}{p_n}}, \quad \text{if } 1 \le p < \infty$$

and

$$\Big\|\sum_{i=1}^m z_i\Big\| = \sup_{n\in\mathbb{N}} \delta_n\ m^{\frac{1}{p_n}}, \quad \text{if } p = 0.$$

On the other hand,

$$\Big\|\sum_{i=1}^m e_i^{(p)}\Big\|^p = m, \quad \text{if } 1 \le p < \infty \quad \text{and} \quad \Big\|\sum_{i=1}^m e_i^{(p)}\Big\| = 1, \quad \text{if } p = 0.$$

Consider the case $1 \le p < \infty$ and put

$$\lambda_m^{(p)} = \frac{\Big\|\sum\limits_{i=1}^m z_i\Big\|^p}{\Big\|\sum\limits_{i=1}^m e_i^{(p)}\Big\|^p} = \sum_{n=1}^\infty \delta_n^p\ m^{\frac{p}{p_n}-1}.$$

If $\inf\limits_n p_n < p$ then there exists an n_0 such that $p_{n_0} < p$ and hence

$$\lambda_m^{(p)} \ge \delta_{n_0}^p\ m^{\frac{p}{p_{n_0}}-1} \to \infty \quad \text{as } m \to \infty.$$

Suppose now that $\inf\limits_n p_n \ge p$. In this case $\frac{p}{p_n} - 1 < 0$ for each n. Given $\varepsilon > 0$, choose n_0 so that $\sum\limits_{n=n_0}^\infty \delta_n^p < \frac{\varepsilon}{2}$. Then choose m_0 so that

$$\big(\max_{1\le i\le n_0} \delta_i\big)^p\, m^{\frac{p}{p_{n_0}}-1} < \frac{\varepsilon}{2n_0}$$

for $m \geq m_0$. Then for such m we have

$$\lambda_m^{(p)} = \sum_{n=1}^{n_0} \delta_n^p \, m^{\frac{p}{p_n} - 1} + \sum_{n=n_0+1}^{\infty} \delta_n^p \, m^{\frac{p}{p_n} - 1}$$

$$\leq \sum_{n=1}^{n_0} \Big(\max_{1 \leq i \leq n_0} \delta_i \Big)^p \, m^{\frac{p}{p_{n_0}} - 1} + \sum_{n=n_0}^{\infty} \delta_n^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The case $p = 0$ is quite trivial: $\lambda_m^{(p)} \to \infty$ as $m \to \infty$ anyway. □

Thus, we have shown that the basis $\{z_i\}_{i=1}^{\infty}$ of Z_p which is normalized and symmetric (by Lemma 2.1) is not equivalent to the unit vector basis of ℓ_p (resp., c_0) (which is also normalized and symmetric), for any value of p. Note that the spaces $\ell_p, \ 1 \leq p < \infty$ and c_0 have, up to equivalence, a unique symmetric basis [6, p.129]. Therefore we obtain

Corollary 4.1. *Let $p \in \{0\} \cup [1, +\infty)$, $\mathcal{P}$ be arbitrary. Then the spaces ℓ_p and Z_p are not isomorphic.*

Of course, for distinct indices $p \neq s$ the spaces ℓ_s and Z_p cannot be isomorphic (see Proposition 4.2 below).

Recall that a linear bounded operator $T : X \to Y$ between Banach spaces (denoted as $T \in \mathcal{L}(X, Y)$ is called compact if $TB(X)$ is a relatively compact set in Y, and is called strictly singular provided the restriction $T|_{X_0}$ of T to any infinite dimensional subspace $X_0 \subseteq X$ is not an isomorphic embedding. Of course, each compact operator is strictly singular, but the converse does not hold, for example for the embedding operators $I_{p,s} : \ell_p \to \ell_s$ when $1 \leq p < s < \infty$.

Two infinite dimensional Banach spaces are said to be totally incomparable if they do not contain isomorphic infinite dimensional subspaces. For example, each two spaces from the class $\{c_0, \ell_p : \ 1 \leq p < \infty\}$ are totally incomparable [6, p.54]. Evidently, if X and Y are totally incomparable and X_1, Y_1 are hereditarily X and respectively, Y then X_1 and Y_1 are totally incomparable too. On the other hand, evidently if X and Y are totally incomparable then each operator $T \in \mathcal{L}(X, Y)$ is strictly singular. Thus we have the following

Proposition 4.2. *Let $s, p \in \{0\} \cup [1, +\infty)$, $X \in \{\ell_s, Z_s\}$ and $Y \in \{\ell_p, Z_p\}$ (if $s = 0$ or $p = 0$ then we mean c_0 instead of ℓ_s or ℓ_p respectively). If $s \neq p$ then every operator $T \in \mathcal{L}(X, Y)$ is strictly singular.*

Now we prove that the Pitt theorem does not hold in general for hereditarily ℓ_p spaces.

Example 4.1. Suppose that $\inf_n p_n \geq s$ where $\mathcal{P}_2 = \{p_1, p_2, \cdots\}$. Then for any p and $\mathcal{P}_1$ there exist non-compact operators

$$T \in \mathcal{L}(\ell_s, Z_p(\mathcal{P}_2)) \quad \text{and} \quad T_1 \in \mathcal{L}(Z_s(\mathcal{P}_1), Z_p(\mathcal{P}_2)).$$

Certainly, the example may be of interest if $s > p$.

Proof. By Theorem 3.1, it is enough to construct a noncompact operator $T \in \mathcal{L}(\ell_s, Z_p)$ where $Z_p = Z_p(\mathcal{P}_2)$. We show that the natural operator from ℓ_s to Z_p which cannot be compact is bounded. Indeed, let $x = \sum_{i=1}^{m} a_i e_i^{(s)} \in \ell_s$. Since $\inf_n p_n \geq s$, then

$$\left(\sum_{i=1}^{m} |a_i|^{p_n} \right)^{\frac{1}{p_n}} \leq \|x\|$$

for each $n \in \mathbb{N}$ and hence by Lemma 2.1

$$\|Tx\| = \left(\sum_{n=1}^{\infty} \delta_n^p \left(\sum_{i=1}^{m} |a_i|^{p_n} \right)^{\frac{p}{p_n}} \right)^{\frac{1}{p}} \leq \|x\|$$

and T can be extended to the whole space ℓ_s. □

5. Remarks and Open Problems

From [7, p.212] we easily deduce

Remark 5.1. If for the set $\mathcal{P}$ we have $1 \leq p \leq \inf_n p_n < p_1 \leq 2$ then the space $X_p^{\mathcal{P}}$ and hence its subspace Z_p is isometric to a subspace of L_s for any $s \in [2, p]$.

We do not know whether the condition $\inf_n p_n \geq s$ is essential in Example 4.1. In a view of Proposition 4.1 (i), it looks very likely. Moreover, note that from a result of H. P. Rosenthal (Theorem A2) [20] and Remark 5.1 we obtain

Corollary 5.1. (1) *Let* $1 \leq p < \cdots < p_2 < p_1 \leq 2 < s < \infty$. *Then every operator* $T \in \mathcal{L}(\ell_s, Z_p)$ *is compact.*

(2) *Let* $1 \leq p < s < \cdots < p_2 < p_1 \leq 2$. *Then every operator* $T \in \mathcal{L}(Z_s, \ell_p)$ *is compact.*

Thus, we have

Problem 1. *Suppose that* $p < s$, $\inf_n p_n < s$ *but the condition in Corollary* 5.1 (*i*) *is not fulfilled. Does there exist a non-compact operator* $T \in \mathcal{L}(\ell_s, Z_p)$*?*

We do not know whether we can replace the range space Z_p by ℓ_p in Example 4.1. More exactly

Problem 2. *Suppose that* $p < s$ *but the condition in Corollary* 5.1 (*ii*) *is not fulfilled. Does there exist a non-compact operator* $T \in \mathcal{L}(Z_s, \ell_p)$*?*

Or, more general

Problem 3. *Let* $1 \le p < s < \infty$ *and let* X *be a hereditarily* ℓ_s *Banach space. Does there exist a non-compact operator* $T \in \mathcal{L}(X, \ell_p)$*?*

We are not interested in the case when the domain space is c_0 because Remark 4 of [20] yields

Remark 5.2. If a Banach space Y contains no subspace isomorphic to c_0 then every operator $T \in \mathcal{L}(c_0, Y)$ is compact.

We would like to ask in general:

Problem 4. *What properties of the spaces* c_0 *and* ℓ_p *for* $1 \le p < \infty$ *remain true for hereditarily* c_0 *and respectively* ℓ_p *spaces and what are not true (otherwise trivial cases)?*

Some more questions concern the geometric structure of the spaces Z_p. Recall that a Banach space X is said to be primary if for every decompositions of X onto complemented subspaces $X = Y \oplus Z$ either Y or Z is isomorphic to X.

Problem 5. *Is* Z_p *primary?*

Problem 6. *How many (finite, countable or uncountable) non-equivalent normalized symmetric bases does* Z_p *have?*

References

[1] P. Azimi, *A new class of Banach sequence spaces* // Bull. Iran. Math. Soc. **28** (2)(2002), 57–68.

[2] P. Azimi and J. N. Hagler, *Examples of hereditarily* ℓ_1 *Banach spaces failing the Schur property* // Pacif. J. Math. **122** (2)(1986), 287–297.

[3] J. Bourgain, ℓ_1*-subspaces of Banach spaces.* Lecture notes. Free University of Brussels.

[4] R. C. James, *A separable somewhat reflexive Banach space with non-separable dual* // Bull. Amer. Math. Soc. **80** (1974), 738–743.

[5] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces.* Handbook of the geometry of Banach spaces. Vol.I. (W. B. Johnson and J. Lindenstrauss eds.), Elsevier, Amsterdam. 2001, 1–84.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I.* Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II.* Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[8] E. Odell and Th. Shlumprecht, *Distortion and asymptotic structure,* Handbook of the geometry of Banach spaces. Vol.II. (W. B. Johnson and J. Lindenstrauss eds.) Elsevier, Amsterdam. 2003, 1333–1360.

[9] M. M. Popov, *A hereditarily ℓ_1 subspace of L_1 without the Schur property* // Proc. Amer. Math. Soc. (2005), (to appear).

[10] H. P. Rosenthal, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$* // J. Funct. Anal. **4** (2)(1969), 176–214.

Contact information

Mikhail M. Popov
Department of Mathematics
Chernivtsi National University
str. Kotsjubyn'skoho 2,
58012 Chernivtsi,
Ukraine
E-Mail: popov@chv.ukrpack.net