Український математичний вісник
Том 4 **(2007)**, № 2, 173 – 179

On 2-primal Ore extensions

Vijay K. Bhat

(*Presented by I. V. Protasov*)

Abstract. Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. We define a δ property on R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where P(R) denotes the prime radical of R. We ultimately show the following.

Let R be a Noetherian δ-ring, which is also an algebra over Q, σ and δ be as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$, P any minimal prime ideal of R. Then $R[x, \sigma, \delta]$ is a 2-primal Noetherian ring.

2000 MSC. 16XX, 16N40, 16P40, 16W20, 16W25.

Key words and phrases. 2-primal, Minimal prime, prime radical, nil radical, automorphism, derivation.

1. Introduction

A ring R always means an associative ring. Q denotes the field of rational numbers. Spec(R) denotes the set of prime ideals of R. MinSpec(R) denotes the sets of minimal prime ideals of R. P(R) and N(R) denote the prime radical and the set of nilpotent elements of R respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

This article concerns the study of ore extensions in terms of 2-primal rings. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [13], Greg Marks discusses the 2-primal property of $R[x, \sigma, \delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$

Received 14.11.2006
Sincere thanks to referee for some suggestions to give the manuscript a modified shape.

ISSN 1810 – 3200.

is a τ-derivation of R. Also let R = K[x], K a field. Then the formal derivative d/dx is a derivation of R.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [10]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring R is 2-primal if and only if nil radical and prime radical of R are same if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$. We also note that a reduced is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [7, 9, 10, 14]. Before proving the main result, we find a relation between the minimal prime ideals of R and those of the Ore extension $R[x, \sigma, \delta]$, where R is a Noetherian Q-algebra, σ is an automorphism of R and δ is a σ-derivation of R. This is proved in Theorem (2.1). Recall that $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a)+\delta(a)$ for all $a \in R$. We take any $f(x) \in R[x, \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x, \sigma, \delta]$ by O(R).

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 3, 4, 8, 11, 12]. Recall that in [11], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies a = 0 for $a \in R$. In [12], Kwak defines a $\sigma(*)$-ring R to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $R[x, \sigma]$.

Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. We introduce a property on R and say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where P(R) denotes the prime radical of R. We note that a ring with identity is not a δ-ring.

Now let R be a Noetherian δ-ring, which is also an algebra over Q such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $R[x, \sigma, \delta]$ is 2-primal. This is proved in Theorem (2.4).

2. Ore extensions

We begin with the following definition:

Definition 2.1. *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-ring if a $\delta(a) \in P(R)$ implies $a \in P(R)$.*

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $I[x, \sigma, \delta]$ is an ideal of $R[x, \sigma, \delta]$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$.

Gabriel proved in Lemma (3.4) of [5] that if R is a Noetherian Q-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. We generalize this for σ-derivation δ of R and give a structure of minimal prime ideals of O(R) in the following Theorem.

Theorem 2.1. *Let R be a Noetherian Q-algebra. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$ and $P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*

Proof. Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Let T = $R[[t, \sigma]]$, the skew power series ring. Now it can be seen that $e^{t\delta}$ is an automorphism of T and $P_1T \in MinSpec(T)$. We also know that $(e^{t\delta})^k(P_1T) \in MinSpec(T)$ for all integers $k \geq 1$. Now T is Noetherian by Exercise (1ZA(c)) of [6], and therefore Theorem (2.4) of [6] implies that MinSpec(T) is finite. So exists an integer an integer $n \geq 1$ such that $(e^{t\delta})^n(P_1T) = P_1T$; i.e. $(e^{nt\delta})(P_1T) = P_1T$. But R is a Q-algebra, therefore, $e^{t\delta}(P_1T) = P_1T$. Now for any $a \in P_1$, $a \in P_1T$ also, and so $e^{t\delta}(a) \in P_1T$; i.e. $a + t\delta(a) + (t^2/2!)\delta^2(a) + \cdots \in P_1T$, which implies that $\delta(a) \in P_1$. Therefore $\delta(P_1) \subseteq P_1$.

Now it can be easily seen that $O(P_1) \in Spec(O(R))$. Suppose that $O(P_1) \notin MinSpec(O(R))$, and $P_2 \subset O(P_1)$ is a minimal prime ideal of O(R). Then we have $P_2 = O(P_2 \cap R) \subset O(P_1) \in MinSpec(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in MinSpec(O(R))$.

Conversely let $P \in MinSpec(O(R))$ with $\sigma(P \cap R) = P \cap R$. Then it can be easily seen that $P \cap R \in Spec(R)$ and $O(P \cap R) \in Spec(O(R))$. Therefore $O(P \cap R) = P$. We now show that $P \cap R \in MinSpec(R)$. Suppose that $P_3 \subset P \cap R$, and $P_3 \in MinSpec(R)$. Then $O(P_3) \subset O(P \cap R) = P$. But $O(P_3) \in Spec(O(R))$ and, $O(P_3) \subset P$, which is not possible. Thus we have $P \cap R \in MinSpec(R)$. □

Proposition 2.1. *Let R be a 2-primal ring. Let σ and δ be as usual such that $\delta(P(R)) \subseteq P(R)$. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. Let $P \in MinSpec(R)$. Now for any $a \in P$, there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of [14]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \in P(R)$; i.e. $\delta(a)\sigma(b)+a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Also $\sigma(P) = P$ and by Proposition (1.11) of [14], P is completely prime, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$. ☐

Theorem 2.2. *Let R be a δ-ring. Let σ and δ be as above such that $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\rho : R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that τ is an automorphism of R/P(R) and ρ is a τ-derivation of R/P(R). Now $a\delta(a) \in P(R)$ if and only if $(a+P(R))\rho(a+P(R)) = P(R)$ in R/P(R). Thus as in Proposition (5) of [8], R is a reduced ring and, therefore R is 2-primal. ☐

Proposition 2.2. *Let R be a ring. Let σ and δ be as usual. Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*
2. *For any completely prime ideal U of $R[x, \sigma, \delta]$, $U \cap R$ is a completely prime ideal of R.*

Proof. **(1)** Let P be a completely prime ideal of R. Now let f(x) = $\sum_{i=0}^{n} x^i a_i \in R[x, \sigma, \delta]$ and g(x) = $\sum_{j=0}^{m} x^j b_j \in R[x, \sigma, \delta]$ be such that $f(x)g(x) \in P[x, \sigma, \delta]$. Suppose $f(x) \notin P[x, \sigma, \delta]$. We will show that $g(x) \in P[x, \sigma, \delta]$. We use induction on n and m. For n = m = 1, the verification is easy. We check for n = 2 and m = 1. Let f(x) = $x^2a+xb+c$ and g(x) = xu + v. Now $f(x)g(x) \in P[x, \sigma, \delta]$ with $f(x) \notin P[x, \sigma, \delta]$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to P or all of them do not belong to P. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in P[x, \sigma, \delta]$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore a, $\delta(a) \in P$. Now $\delta(a)u+\sigma(b)u+av \in P$ implies that $\sigma(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore we have $u \in P$. Now $\delta(b)u + \sigma(c)u + bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus we have $g(x) \in P[x, \sigma, \delta]$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above a, $\delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; i.e. $b, \delta(b) \in P$. Also $\delta(b)u+\sigma(c)u+bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore $g(x) \in P[x, \sigma, \delta]$.

Now suppose the result is true for k, $n = k > 2$ and m = 1. We will prove for n = k+1. Let f(x) = $x^{k+1}a_{k+1} + x^k a_k + \cdots + xa_1 + a_0$, and g(x) = $xb_1 + b_0$ be such that $f(x)g(x) \in P[x, \sigma, \delta]$, but $f(x) \notin P[x, \sigma, \delta]$. We will show that $g(x) \in P[x, \sigma, \delta]$. If $a_{k+1} \notin P$, then equating coefficients of x^{k+2}, we get $\sigma(a_{k+1})b_1 \in P$, which implies that $b_1 \in P$. Now equating coefficients of x^{k+1}, we get $\sigma(a_k)b_1 + a_{k+1}b_0 \in P$, which implies that $a_{k+1}b_0 \in P$, and therefore $b_0 \in P$. Hence $g(x) \in P[x, \sigma, \delta]$.

If $a_j \notin P$, $0 \leq j \leq k$, then using induction hypothesis, we get that $g(x) \in P[x, \sigma, \delta]$. Therefore the statement is true for all n. Now using the same process, it can be easily seen that the statement is true for all m also. The details are left to the reader.

(2) Let U be a completely prime ideal of $R[x, \sigma, \delta]$. Suppose $a, b \in R$ are such that $ab \in U \cap R$ with $a \notin U \cap R$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in U \cap R \subseteq U$, with $a \notin U$. Therefore we have $b \in U$, and thus $b \in U \cap R$. □

Corollary 2.1. *Let R be a δ-ring, where σ and δ as usual such that $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*

Proof. R is 2-primal by Theorem (2.2), and so by Proposition (2.1) $\delta(P) \subseteq P$. Further more P is a completely prime ideal of R by Proposition (1.11) of [10]. Now use Proposition (2.2). □

We now prove the following Theorem, which is crucial in proving Theorem 2.4.

Theorem 2.3. *Let R be a δ-ring, where σ and δ as usual such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $R[x, \sigma, \delta]$ is 2-primal if and only if $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.*

Proof. Let $R[x, \sigma, \delta]$ be 2-primal. Now by Corollary (2.1) $P(R[x, \sigma, \delta]) \subseteq P(R)[x, \sigma, \delta]$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in P(R)[x, \sigma, \delta]$. Now R is a 2-primal subring of $R[x, \sigma, \delta]$ by Theorem (2.2), which implies that a_j is nilpotent and thus $a_j \in N(R[x, \sigma, \delta]) = P(R[x, \sigma, \delta])$, and so we have $x^j a_j \in P(R[x, \sigma, \delta])$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(R[x, \sigma, \delta])$. Hence $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.

Conversely suppose $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. We will show that $R[x, \sigma, \delta]$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in R[x, \sigma, \delta]$, $b_n \neq 0$, be such that $(g(x))^2 \in P(R[x, \sigma, \delta]) = P(R)[x, \sigma, \delta]$. We will show that $g(x) \in P(R[x, \sigma, \delta])$. Now leading coefficient $\sigma^{2n-1}(a_n)a_n \in P(R) \subseteq P$, for all

$P \in MinSpec(R)$. Now $\sigma(P) = P$ and P is completely prime by Proposition (1.11) of [10]. Therefore we have $a_n \in P$, for all $P \in MinSpec(R)$; i.e. $a_n \in P(R)$. Now since $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$, we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x,\sigma,\delta]) = P(R)[x,\sigma,\delta]$ and as above we get $a_{n-1} \in P(R)$. With the same process in a finite number of steps we get $a_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in P(R)[x,\sigma,\delta]$; i.e. $g(x) \in P(R[x,\sigma,\delta])$. Therefore $P(R[x,\sigma,\delta])$ is completely semiprime. Hence $R[x,\sigma,\delta]$ is 2-primal. □

Theorem 2.4. *Let R be a Noetherian δ-ring, which is also an algebra over Q such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$, where σ and δ are as usual. Then $R[x,\sigma,\delta]$ is 2-primal.*

Proof. We use Theorem (2.1) to get that $P(R)[x,\sigma,\delta] = P(R[x,\sigma,\delta])$, and now the result is obvious by using Theorem (2.3). □

Corollary 2.2. *Let R be a commutative Noetherian δ-ring, which is also an algebra over Q such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$, where σ and δ are as usual. Then $R[x,\sigma,\delta]$ is 2-primal.*

Proof. Using Theorem (1) of [15] we get $\delta(P(R)) \subseteq P(R)$. Now rest is obvious. □

The above gives rise to the following questions:

If R is a Noetherian Q-algebra (even commutative), σ is an automorphism of R and δ is a σ-derivation of R. Is $R[x,\sigma,\delta]$ 2-primal? The main problem is to get Theorem (2.3) satisfied.

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Contact information

Vijay Kumar Bhat
School of Applied Physics and Mathematics,
SMVD University,
P/o Kakryal, Udhampur, J and K,
India 182121
E-Mail: vijaykumarbhat2000@yahoo.com