ASYMPTOTIC EXPANSIONS FOR EIGENVALUES AND EIGENFUNCTIONS OF ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH RAPIDLY OSCILLATING COEFFICIENTS

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The problem of the construction of asymptotic expansions for solutions of such spectral problems stands in the homogenization theory [1–4] for about 10 years—since such expansions were constructed at first for the Sturm-Liouville problem [5]. But the method proposed worked only for the case of a simple spectrum of the corresponding homogenized problem. At the time when the main trouble for boundary-value problems with rapidly oscillating coefficients was a construction of a boundary layer near the boundary the main trouble of the corresponding spectral problems was multiplicity of the spectrum. The only progress in this direction was an approach of [6] in which the author overcame the both obstacles by imposing certain symmetry relations on the domain and on the coefficients of the equation. It is important to notice that all the troubles are arising namely in the process of formal construction of expansions and not in there justification. The methods of the justification are worked out well (see [4], [6]).

In the paper I solve the problem of a multiple spectrum without any artificial conditions imposed and obtain whole asymptotic expansions for eigenvalues and eigenfunctions of the general smooth boundary-value problem for an elliptic operator with non-uniformly oscillating coefficients (whose homogenization is made in [7]). I assume that the boundary layers (the functions $N^2_{p,\alpha}$ in sequel) are constructed.

In what follows we denote $D_i = \partial/\partial\xi_i$; $\langle f\rangle = \int_{(0;1)^d} f(\xi)d\xi$; $D^\alpha = \partial^{|\alpha|}/\partial x_1^{|\alpha_1|},\ldots, x_d^{|\alpha_d|}$, $|\alpha| = |\alpha_1|+\cdots+|\alpha_d|$ for $\alpha = (\alpha_1,\ldots,\alpha_d)$, and by repeated indices i, j we assume the summation from 1 to d; $d\ge 1$ is the dimension of our space.

In a bounded domain $\Omega\subset\mathbb{R}^d$, $d\ge 1$, consider the problem

$$-\frac{\partial}{\partial x_i}\left(A_{ij}\left(x,\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}u_\varepsilon(x)\right)+A\left(x,\frac{x}{\varepsilon}\right)u_\varepsilon(x) = \lambda_\varepsilon\rho\left(x,\frac{x}{\varepsilon}\right)u_\varepsilon(x),\quad x\in\Omega, \tag{1}$$

$$\mathcal{B}_\varepsilon u_\varepsilon = 0, \tag{2}$$

where A, ρ, A_{ij}, $1\le i\le d$, $1\le j\le d$, are 1-periodic with respect to $\xi = x/\varepsilon$ and for all $x,\xi,\eta\in\mathbb{R}^d$

$$\rho(x,\xi)>0,\quad A(x,\xi)>0,\quad A_{ij}(x,\xi)\eta_i\eta_j\ge\kappa|\eta|^2,$$
$$A_{ij}(x,\xi) = A_{ji}(x,\xi)\quad\text{for all}\quad 1\le i\le d,\quad 1\le j\le d,$$

where $\kappa > 0$ does not depend on x, ξ, η; $\mathcal{B}_\varepsilon$ is a boundary condition operator (Dirichlet, Neumann and 1-periodic boundary conditions mixed); $0 < \varepsilon \ll 1$; all the elements of the problem are infinitely smooth (in the case of 1-periodic boundary condition with respect to $x_{l+1}, \dots, x_d$, $0 \le l < d$, we suppose that $\Omega \subset \mathbb{R}^l \times (0;1)^{d-l}$; $\{x \in \overline{\Omega} : x_i = 0\} + \mathbf{e}_i = \{x \in \overline{\Omega} : x_i = 1\}$, $l < i \le d$; not Ω but $\overline{\Omega} + \{0\}^l \times \mathbb{Z}^{d-l}$ has a smooth boundary; other boundary conditions are adapted).

Let us suppose that for each sequence $\Lambda = (\lambda_0, \lambda_1, \dots)$ we have constructed such functions $N_{p,\alpha}(x,\xi)$, $0 \le |\alpha| \le p$, on $\Omega \times \varepsilon^{-1}\Omega$ that the formal substitutions

$$\lambda_\varepsilon = \sum_{p=0}^{+\infty} \varepsilon^p \lambda_p, \qquad v_\varepsilon(x) = \sum_{p=0}^{+\infty} \varepsilon^p v_p(x),$$
$$u_\varepsilon(x) = \sum_{p=0}^{+\infty} \varepsilon^p \sum_{|\alpha| \le p} N_{p,\alpha}\left(x, \frac{x}{\varepsilon}\right) D^\alpha v_\varepsilon(x), \qquad x \in \Omega,$$

transform the input problem (1), (2) into the sequence of problems

$$\sum_{n=0}^{p} L_{p-n} v_n(x) = 0, \quad x \in \Omega, \tag{3}$$
$$\sum_{n=0}^{p} B_{p-n} v_n = 0; \quad p \ge 0, \tag{4}$$

where differential operators L_p and boundary condition operators B_p, $p \ge 0$, do not depend on ε. The structure of the functions $N_{p,\alpha}$ is the following: $N_{p,\alpha} = N^1_{p,\alpha} + N^2_{p,\alpha}$, where $N^1_{p,\alpha}$ are the 1-periodic components, $N^2_{p,\alpha}$ are the boundary layer functions.

The functions $N^1_{p,\alpha}(x,\xi)$ are defined on $\Omega \times \mathbb{R}^d$ and are 1-periodic with respect to ξ. They are determined by the equation (1) only and determine the operators L_p by the formulae

$$\begin{aligned}
&N^1_{0,0} = 1;\\
&-D_i(A_{ij}(x,\xi) D_j N^1_{p,\alpha})(x,\xi) = \hat{h}_{p,\alpha}(x) - h_{p,\alpha}(x,\xi), \quad \xi \in \mathbb{R}^d,\\
&\left\langle N^1_{p,\alpha}(x,\cdot)\right\rangle = 0,\\
&h_{p,\alpha} = -(D_i(A_{ij} N^1_{p-1,\alpha-\delta_j}) + A_{ij} D_i N^1_{p-1,\alpha-\delta_j} +\\
&\qquad + A_{ij} N^1_{p-2,\alpha-\delta_i-\delta_j}) + A N^1_{p-2,\alpha} - \sum_{t=|\alpha|}^{p-2} \lambda_{p-2-t} B N^1_{t,\alpha}\ ,\\
&\hat{h}_{p,\alpha}(x) = \langle h_{p,\alpha}(x,\cdot)\rangle\,, \quad p > 0, \quad |\alpha| \le p, \quad x \in \Omega;\\
&L_p = \sum_{|\alpha| \le p+2} \hat{h}_{p+2,\alpha}(x) D^\alpha, \quad p \ge 0.
\end{aligned}$$

Note that $L_p = \dot{L}_p - \lambda_p \langle B\rangle$ where operator $\dot{L}_p$ does not depend on λ_k, $k \ge p$; $p \ge 0$.

Proposition 1. *All the operators L_p, $p \geq 0$, are self-adjoint.*

This fact was not proved for $p \geq 1$; the self-adjointness of L_0 is proved in [7].

Remark. Proposition 1 is also true for boundary-value (not spectrum) problems irrespective of Ω and $\mathcal{B}_\varepsilon$. So the volume of calculations when computing the operators L_p, $p \geq 1$, decreases twice.

The functions $N^2_{p,\alpha}(x,\xi)$ tend to zero exponentially when moving from the boundary, they satisfy the equations

$$-D_i(A_{ij}D_jN^2_{p,\alpha}) = -(D_i(A_{ij}N^2_{p-1,\alpha-\delta_j}) + A_{ij}D_iN^2_{p-1,\alpha-\delta_j} + A_{ij}N^2_{p-2,\alpha-\delta_i-\delta_j}) + AN^2_{p-2,\alpha} - \sum_{t=|\alpha|}^{p-2} \lambda_{p-2-t}BN^2_{t,\alpha}.$$

These functions determine the operators B_p and are constructed only for a few domains and boundary conditions. Always $N^2_{0,0} = 0$.

The homogenized problem is the problem for $p = 0$ from the system (3), (4). Namely the system (3), (4) was not solved in the case of a multiple spectrum of the homogenized problem. The next result shows some symmetry in this system and opens a way to its solving.

Lemma. *Let the functions $v_0, \dots, v_k$ and $\tilde{v}_0, \dots, \tilde{v}_k$ satisfy equations (3) for $p < k$ and boundary conditions (4) for $p \leq k$ with some fixed $\lambda_0, \dots, \lambda_{k-1} \in \mathbb{R}$ and $k \geq 1$. Then*

$$\int_\Omega \left(\sum_{n=1}^{k} L_{k-n}v_n + \dot{L}_k v_0 \right) \tilde{v}_0 dx = \int_\Omega \left(\sum_{n=1}^{k} L_{k-n}\tilde{v}_n + \dot{L}_k \tilde{v}_0 \right) v_0 dx. \tag{5}$$

Denotation. *Let $H_{\lambda_0,\dots,\lambda_k}$ be the set of such functions v_0 that the part of the system (3), (4) for $p \leq k$ has a solution with these $\lambda_0, \dots, \lambda_k \in \mathbb{R}$, $k \geq 0$.*

All these sets are finite-dimensional subspaces of $C^\infty(\Omega)$. In particular, $H_{\lambda_0}\backslash\{0\}$ is the set of all the eigenfunctions corresponding to the eigenvalue λ_0 of the homogenized problem. Always

$$H_{\lambda_0,\dots,\lambda_k} \subset H_{\lambda_0,\dots,\lambda_{k-1}} \subset \cdots \subset H_{\lambda_0} \subset C^\infty(\Omega).$$

In accordance with Lemma (for $k+1$ instead of k) the expression

$$(\mathcal{P}_{\lambda_0,\dots,\lambda_k} v_0, \tilde{v}_0) = \lambda_0 \int_\Omega \left(\sum_{n=1}^{k+1} L_{k+1-n}v_n + \dot{L}_{k+1} v_0 \right) \tilde{v}_0 dx$$

determines a self-adjoint operator $\mathcal{P}_{\lambda_0,\dots,\lambda_k}$ in $H_{\lambda_0,\dots,\lambda_k}$, here the scalar product

$$(u, v) = \int_\Omega (\dot{L}_0 u) v dx \tag{6}$$

is equivalent to the standard one in Sobolev space $W^1_2(\Omega)$.

Proposition 2. *Let* $H_{\lambda_0,\dots,\lambda_k} \neq \{0\}$ *for some* $\lambda_0,\dots,\lambda_k \in \mathbb{R}$, $k \geq 0$. *Then the general solution of the part of the system (3), (4) for* $p \leq k$ *we get in the form:* $v_m \in v_m^* + H_{\lambda_0,\dots,\lambda_{k-m}}$, $0 \leq m \leq k$, *where* $v_0^* = 0$ *and* v_m^* *is determining algorithmically by already chosen* $v_0,\dots,v_{m-1}$ *for* $1 \leq m \leq k$.

Proceeding to the next problem from (3), (4) (for $p = k+1$*) we get a decomposition* $H_{\lambda_0,\dots,\lambda_k} = \bigoplus_{\lambda_{k+1}\in S_{\lambda_0,\dots,\lambda_k}} H_{\lambda_0,\dots,\lambda_{k+1}}$, *where* $S_{\lambda_0,\dots,\lambda_k}$ *is a spectrum,* $H_{\lambda_0,\dots,\lambda_{k+1}}$ *is a subspace of the eigenfunctions of* $\mathcal{P}_{\lambda_0,\dots,\lambda_k}$ *corresponding to the eigenvalue* λ_{k+1}.

As a result we obtain a sequence of sequences $\Lambda^{(1)} \leq \Lambda^{(2)} \leq \dots$ in correspondence with the sequence ${\lambda_0}^{(1)} \leq {\lambda_0}^{(2)} \leq \dots$ of eigenvalues of the homogenized problem (they are taken with an account of a multiplicity) and get the corresponding decomposition $\{u \in W_2^1(\Omega) : B_0 u = 0\} = \bigoplus_{n=1}^{+\infty} H_{s_n}$, where $s_1 = 1$, $\Lambda^{(s)} = \Lambda^{(s_n)}$ and $H_s = H_{s_n}$ for $s_n \leq s < s_{n+1}$, $s_{n+1} = s_n + \dim H_{s_n}$, $n \geq 1$, and $H_s = \bigcap_{k=0}^{+\infty} H_{{\lambda_0}^{(s)},\dots,{\lambda_k}^{(s)}}$, $s \geq 1$.

Let the eigenvalues of the input problem (1), (2) be enumerated too (with an account of a multiplicity): $\lambda_\varepsilon^{(1)} \leq \lambda_\varepsilon^{(2)} \leq \dots$ The justification procedure (see it in [6] for different problems) gives the next result.

Theorem. *For every* $s \geq 1$ *we have*

$$\lambda_\varepsilon^{(s)} \approx \sum_{p=0}^{+\infty} \varepsilon^p \lambda_p^{(s)}, \quad \varepsilon \to 0.$$

For each $r \geq 0$ *and for the functions* $v_0,\dots,v_r$ *satisfying the part of the system (3), (4) for* $p \leq r$ *with* ${\lambda_0}^{(s)},\dots,\lambda_r^{(s)}$ *we have*

$$\|u_\varepsilon - \sum_{p=0}^{r} \varepsilon^p \sum_{n=0}^{p} \sum_{|\alpha|\leq p-n} N_{p-n,\alpha}\left(\cdot, \frac{\cdot}{\varepsilon}\right) D^\alpha v_n\| = O(\varepsilon^r), \quad \varepsilon \to 0,$$

here u_ε *is a linear combination of the eigenfunctions of the problem (1), (2) corresponding to the eigenvalues* $\lambda_\varepsilon^{(t)}$ *for* $s_n \leq t < s_{n+1}$ *(where* $s_n \leq s < s_{n+1}$*); the norm is corresponding to the introduced scalar product.*

References

1. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, Northen Holland, Amsterdam, 1978.
2. L. Tartar, *Homogénéisation: Cours Peccot*, Collège de France, Paris, 1977.
3. N. S. Bakhvalov, and G. P. Panasenko, *Homogenization of Processes in Periodic Media*, Nauka, Moscow, 1981. (in Russian)
4. O. A. Oleinik, G. A. Yosifian, and A. S. Shamaev, *Mathematical Problems of Theory of Strongly Non-Homogenious Ellastic Media*, Moscow Univ. Publ., Moscow, 1990. (in Russian)
5. G. A. Yosifian, O. A. Oleinik, and A. S. Shamaev, *Asymptotical expansions for eigenvalues and eigenfunctions of Sturm-Liouville problem with rapidly oscillating coefficients*, Vestnik Moscovskogo Universiteta. Ser. math.-mech (1985), no. 6, 22–35. (in Russian)
6. T. A. Melnik, *Asymptotic expansions of eigenvalues and eigenfunctions for elliptic boundary-value problems with rapidly oscillating coefficients in a perforated cube*, Trudy Seminara imeni G. P. Petrovskogo **17** (1994); English translation in Jour. of Math. Sciences **75** (1995), no. 3.

7. O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian, *Homogenization of eigenvalues and eigenfunctions of the boundary value problems in perforated domains for elliptic equations with non-uniformly oscillating coefficients*, in Current Topics in Partial Differential Equations, Kinokuniya Co., Tokyo, 1986, pp. 187–216.

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Kamyanets-Podilsky,
Kosmonavtiv str., 2, 48.