NONLINEAR PROBLEMS OF HEAT RADIATING BODY WITH THERMAL THIN COVER

© A.A. Berezovsky, G.N. Komarov, O.G. Nartova

Abstract

We consider a boundary initial problem for heat conductivity with nonlinear boundary condition, which contains time derivative and tangential part of Laplace operator. Existence and uniqueness theorems for positive solutions of mentioned problems are proved.

The determining of temperature field $u(P,t) > 0$ of a heat radiating body Ω, partially or fully covered by thin layer ω, reduced to solution of next initial boundary problem for conjugation:

$$\begin{gathered}
\operatorname{div}(\lambda \operatorname{grad} u) - c\rho u_t = -w, \quad P \in \Omega \cup \omega, \ t > 0, \\
u(P,0) = u_0(P), \quad P \in \overline{\Omega \cup \omega}, \\
\lambda \frac{\partial u}{\partial n} + \alpha (u - u_c) = 0, \quad P \in S_1, \ t > 0, \\
\lambda \frac{\partial u}{\partial n} + \sigma\varepsilon (u^4 - u_c^4) = 0, \quad P \in \partial\omega_+, \ t > 0, \\
[u]_\Phi = 0, \quad \left[\lambda \frac{\partial u}{\partial n}\right]_\Phi = 0.
\end{gathered} \tag{1}$$

Here $\lambda, c \ \ \rho$ are coefficients of heat conductivity, thermal heat capacity and density, that are piecewise constant functions for domains Ω and ω; w is heat source; u_0 is initial temperature distribution; α is coefficient of heat transfer, u_c is temperature of environment, σ Stefan-Boltsman constant, ε is the degree of blackness for covering surface $\partial\omega_+$; $S = S_1 \cup S_2$ is a surface that bounded body Ω; the mark $[\cdot]$ means a saltus of value in brackets for transition through surface S_2, which can be described by equation $\Phi(P) = 0$. In particular, the surface S_2 can coincide with a whole surface S, and coefficient of heat transfer $\alpha = \infty$.

Analyzing for domain ω differential equation (1) of the problem in normal-tangential form with respect to initial point $P \in \Phi$, we obtain

$$\begin{gathered}
\frac{\partial}{\partial n}\left(\lambda \frac{\partial u}{\partial n}\right) + \operatorname{div}_\tau(\lambda \operatorname{grad}_\tau u) - c\rho u_t = -w, \\
\xi, \eta, n \in \omega, \quad t > 0,
\end{gathered} \tag{2}$$

where

$$\begin{gathered}
\operatorname{div}\vec{A} = \operatorname{div}_\tau \vec{A}_\tau + \frac{\partial A_n}{\partial n}, \quad \operatorname{grad}\varphi = \operatorname{grad}_\tau \varphi + \vec{n}\,\frac{\partial \varphi}{\partial n}, \\
\operatorname{div}_\tau \vec{A}_\tau = \frac{\partial A_\xi}{\partial \xi} + \frac{\partial A_\eta}{\partial \eta}, \quad \operatorname{grad}_\tau \varphi = \vec{i}\,\frac{\partial \varphi}{\partial \xi} + \vec{j}\,\frac{\partial \varphi}{\partial \eta},
\end{gathered} \tag{3}$$

$A_\xi,\ A_\eta,\ A_n$ are vector $\vec{A}$ components.

If we average equation (2) with respect to covering thickness, with accounting boundary condition on $\partial\omega$, second conjugate condition on S_2 and identifying average value of temperature $\bar{u}$ with its magnitude on S_2, we obtain impedance boundary condition

$$\lambda_- \frac{\partial u}{\partial n}\Big|_{\Phi-0} - d\mathrm{div}_\tau(\lambda_+ \mathrm{grad}_\tau u) + c_+\rho_+ d u_t + \sigma\varepsilon(u^4 - u_c^4) = d\bar{w}, \qquad \xi,\eta,n \in \Phi,\ t>0, \tag{4}$$

where $d = d(\xi,\eta)$ is the thickness of cover; indexes + and - corresponds to body Ω and its covering ω; the dash means averaging across thickness d.

After transition to a new time variable and parameters $\tau = \lambda_- t / c_- \rho_-$ and $h = \alpha/\lambda_+,\ \beta = \lambda_- d/\lambda_+, \kappa = \sigma\varepsilon/\lambda_-, \gamma = c_+\rho_+ d/c_-\rho_-,$
$f = \bar{w}/\lambda_-,\ q = \kappa u_c^4 + d\bar{w}/\lambda_-$, the problem reduced to canonical form

$$\begin{gathered} \Delta u - u_t = -f, \quad P\in\Omega,\ \ t>0, \\ u(P,0) = u_0(P), \quad P\in\bar{\Omega}, \\ \frac{\partial u}{\partial n} + h(u-u_c) = 0,\ \ P\in S_1,\ t>0, \\ \frac{\partial u}{\partial n} - \beta\Delta_\tau u + \gamma u_t + \kappa u^4 = q,\ \ P\in S_2,\ t>0, \end{gathered} \tag{5}$$

where previous notation t is conserved for new variable τ.

Theorem 1. *If for any initial data and parameters there is a positive solution of the problem* (5) *then this solution is unique.*

Proof. Let's $u_1(P,t)$ and $u_2(P,t)$ two different positive solutions of the problem (5). Then we shall obtain next initial boundary problem for difference $u(P,t) = u_2(P,t) - u_1(P,t)$:

$$\begin{gathered} \Delta u - u_t = 0, \quad P\in\Omega,\ \ t>0, \\ u(P,0) = 0, \quad P\in\bar{\Omega}, \\ \frac{\partial u}{\partial n} + hu = 0,\ \ P\in S_1,\ t>0, \\ \frac{\partial u}{\partial n} - \beta\Delta_\tau u + \gamma u_t + \kappa(u_2^4 - u_1^4) = 0,\ \ P\in S_2,\ t>0. \end{gathered} \tag{6}$$

After multiplying differential equation of the problem (6) by $u(P,t)$ and integration for domain Ω with accounting first Green formula

$$\int\limits_\Omega u\Delta u dv = \int\limits_{S_1} u\frac{\partial u}{\partial n} ds + \int\limits_{S_2} u\frac{\partial u}{\partial n} ds - \int\limits_\Omega (\mathrm{grad} u)^2 dv,$$

we obtain

$$\int\limits_\Omega u u_t dv = \int\limits_{S1} u\frac{\partial u}{\partial n} ds + \int\limits_{S_2} u\frac{\partial u}{\partial n} ds - \int\limits_\Omega (\mathrm{grad} u)^2 dv. \tag{7}$$

With accounting boundary conditions integrals on S_1 and S_2 can be transform to

$$\int\limits_{S_1} u\frac{\partial u}{\partial n} ds = -h\int\limits_{S_1} u^2 ds, \tag{8}$$

$$\int\limits_{S_2} u\frac{\partial u}{\partial n}ds = \int\limits_{S_2} \beta u \Delta_\tau u ds - \int\limits_{S_2} \gamma u u_\tau ds - \kappa \int\limits_{S_2} u(u_2^4 - u_1^4)ds. \tag{9}$$

Let's consider the first integral from right side of formula (9). According to a formula of vector analysis

$$\beta u \Delta_\tau u = \mathrm{div}_\tau(\beta u \mathrm{grad}_\tau u) - (\mathrm{grad}_\tau \beta u, \mathrm{grad}_\tau u)$$

and Gausse - Ostrogradsky formula it can be transform to next form

$$\int\limits_{S_2} \beta u \Delta_\tau u ds = \oint\limits_{L} \beta u \frac{\partial u}{\partial n_\perp} dl - \int\limits_{S_2} \beta (\mathrm{grad} u)^2 ds,$$

where $\vec{n}_\perp$ is unit vector directed towards normal of curve L.

The integral for contour L is equal zero if

$$S_2 = S, \ \ \mathrm{mes} S_1 = 0$$

and if d, and so β, is equal zero on L, i.e. the cover vanished on contour L, as well as for first boundary condition, when$h = \infty$.

For these cases

$$\int\limits_{S_2} \beta u \Delta_\tau u ds = - \int\limits_{S_2} \beta (\mathrm{grad} u)^2 ds,$$

we can rewrite equation (7) in the form

$$\begin{gathered} \frac{dI}{dt} = - \int\limits_{\Omega} (\mathrm{grad} u)^2 dv - h \int\limits_{S_1} u^2 ds - \\ \int\limits_{S_2} \beta (\mathrm{grad}_\tau u)^2 ds - \kappa \int\limits_{S_2} u(u_2^4 - u_1^4) ds, \end{gathered} \tag{10}$$

where

$$I = \frac{1}{2} \int\limits_{\Omega} (u^2)_t dv + \frac{1}{2} \int\limits_{S_2} \gamma (u^2)_t ds. \tag{11}$$

Since

$$u(u_2^4 - u_1^4) = (u_2 - u_1)^2 (u_2^3 + u_2^2 u_1 + u_2 u_1^2 + u_1^3) \geq 0,$$

it is evident that $\frac{dI}{dt} \leq 0$. But since by virtue of initial condition $I = 0$ for $t = 0$, so $I \leq 0$ for all $t \geq 0$. From the other side according (11) $I \geq 0$. The solution of such contradiction can be find only for $I = 0 \Rightarrow u_2 - u_1 = 0$, $P \in \Omega \cup S_2$, That is the proof of the theorem.

For heat conductivity equation operator the second Green formula takes place

$$\begin{gathered} \int\limits_0^{\tau+0} \int\limits_{\Omega} [v(\Delta u - u_t) - u(\Delta v + v_t)\ ] dv_P dt = \\ = \int\limits_0^{\tau+0} \oint\limits_{S} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds_P dt - \int\limits_{\Omega} vu|_0^{\tau+0} dv_P. \end{gathered} \tag{12}$$

Let's transform the first integral of right side of this equality to the form that contain linear operators of boundary conditions of the problem (5). It is simple to detect that

$$\begin{gathered}\int_0^{\tau+0}\oint_S\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds_P dt=\\ \int_0^{\tau+0}\int_{S_1}\left[v\left(\frac{\partial u}{\partial n}+hu\right)-u\left(\frac{\partial v}{\partial n}+hv\right)\right]ds_P dt+\\ +\int_0^{\tau+0}\int_{S_2}\left[v\left(\frac{\partial u}{\partial n}-\beta\Delta_\tau u+\gamma u_t\right)-\right.\\ \left.-u\left(\frac{\partial v}{\partial n}-\beta\Delta_\tau v-\gamma v_t\right)\right]ds_P dt-\\ -\int_{S_2}\gamma vu|_0^{\tau+0}\,ds_P+\int_0^{\tau+0}\oint_L\beta\left(v\frac{\partial u}{\partial n_\perp}-u\frac{\partial v}{\partial n_\perp}\right)dl_P dt.\end{gathered}\tag{13}$$

Further on we restrict in studying Dirichlet conditions onS_1, for $h=\infty$. The second Green formula can be transform to

$$\begin{gathered}\int_0^{\tau+0}\int_\Omega\left[v(\Delta u-u_t)-u(\Delta v+v_t)\ \right]dv_P dt=\int_0^{\tau+0}\int_{S_1}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds_P dt+\\ +\int_0^{\tau+0}\int_{S_2}\left[v\left(\frac{\partial u}{\partial n}-\beta\Delta_\tau u+\gamma u_t\right)-u\left(\frac{\partial v}{\partial n}-\beta\Delta_\tau v-\gamma v_t\right)\right]ds_P dt-\\ -\int_\Omega vu|_0^{\tau+0}\,dv_P-\int_{S_2}\gamma vu|_0^{\tau+0}\,ds_P+\int_0^{\tau+0}\oint_L\beta\left(v\frac{\partial u}{\partial n_\perp}-u\frac{\partial v}{\partial n_\perp}\right)dl_P dt.\end{gathered}\tag{14}$$

Let's introduce Green function $G(P,Q;t-\tau)$, as a solution of next initial boundary problem:

$$\begin{gathered}\Delta_P G+G_t=-\delta(P-Q)\delta(t-\tau),\quad P,Q\in\Omega,\ t>0,\\ G(P,Q;t-\tau)=0,\quad t>\tau P,Q\in\Omega\cup S,\\ G(P,Q;t-\tau)=0,\quad P\in S_1,\ Q\in\Omega,\\ \frac{\partial G}{\partial n_P}-\beta\Delta_{\tau P}G-\gamma G_t=0,\quad P\in S_2,\ Q\in\Omega,\end{gathered}\tag{15}$$

where $\delta(P-Q)$ $\delta(t-\tau)$ is Dirack delta-function for points $P=Q$ and moment $t=\tau$.

By using Green formula (14) and introduced Green function the solution of initial boundary problem (5) can be transform to solution of nonlinear integral equation of least dimension.

$$\begin{gathered}u(Q,\tau)=u_l(Q,\tau)-\kappa\int_0^\tau\int_{S_2}G(P,Q;t-\tau)u^4(P,t)ds_P dt,\\ Q\in S_2,\quad \tau>0,\end{gathered}\tag{16}$$

where

$$u_l(Q,\tau)=((G(P,Q;0-\tau),u_0(P)))+\int_0^\tau\int_\Omega G(P,Q;t-\tau)f(P,t)dv_P dt+$$

$$+\int_0^{\tau}\int_{S_2} u_c(P,t)\frac{\partial G}{\partial n_P}ds_P dt+\kappa\int_0^{\tau}\int_{S_2} G(P,Q;t-\tau)q(P,t)ds_P dt+ \tag{17}$$

$$+\int_0^{\tau}\oint_L u_c(P,t)\frac{\partial G}{\partial n_\perp}dl_P dt,\quad Q\in\Omega\cup S,\ \tau>0,$$

and double brackets denote scalar product

$$((u,v))=\int_\Omega u(P)v(P)dv_P+\int_{S_2}\gamma u(P)v(P)ds_P, \tag{18}$$

which generate norm

$$\langle\langle u\rangle\rangle^2=\int_\Omega u^2 dv_P+\int_{S_2}\gamma u^2(P)ds_P. \tag{19}$$

The temperature field in body Ω is determined as quadrature (16) with using solution of integral equation (16) when $Q\in\Omega\cup S$. Let's note that equation (16) is the equation of least dimension, because we should determine only temperature of the surface S_2. It presents a nonlinear integral equation of Hammershtain type with respect to spatial variable and Volterra type for time [1,2]. The solution of such equation can be obtained by successive approximation method. At that next theorem will take place.

Theorem 2. *If Green function* $G(P,Q;t-\tau)\geq 0$ *and the solution of corresponding linear problem* $u_l(P,t)\geq 0$ *is such, that*

$$\begin{gathered}\int_0^T d\tau\int_0^{\tau} dt\int_{S_2} ds_P\int_{S_2} G_2(P,Q;t-\tau)ds_Q<\infty,\\ \int_0^T dt\int_{S_2} u_l^8(P,t)ds_P<\infty;\end{gathered} \tag{20}$$

$$u_l(Q,\tau)\geq\kappa\int_0^{\tau} dt\int_{S_2} G(P,Q;t-\tau)u_l^4(P,t)ds_P, \tag{21}$$

for main domain $0\leq t\leq\tau\leq T;\ P,\ Q\in S_2$, *then the positive solution of integral equation (16) exists and is unique.*

For **proving** this theorem let's apply method of successive approximations

$$\begin{gathered}u_n(Q,\tau)=u_l(Q,\tau)-\kappa\int_0^{\tau}\int_{S_2} G(P,Q;t-\tau)u_{n-1}^4(P,t)ds_P dt,\\ n=1,2,...,\\ u_0(Q,\tau)=u_l(Q,\tau).\end{gathered} \tag{22}$$

By virtue of (21) it is evident that

$$\begin{gathered}0\leq u_1(Q,\tau)\leq u_3(Q,\tau)\leq ...\leq u_{2n+1}(Q,\tau)\leq ...\leq u(Q,\tau),\\ u_2(Q,\tau)\geq u_4(Q,\tau)\geq ...\geq u_{2n}(Q,\tau)\geq ...\geq u(Q,\tau).\end{gathered} \tag{23}$$

Accounting (20) we obtain

$$|u_1(Q,\tau)-u_0(Q,\tau)| = \\ = \kappa\left|\int\limits_0^{\tau}\int\limits_{S_2} G(P,Q;t-\tau)u_l^4(P,t)ds_P dt\right| \le n(Q,\tau), \tag{24}$$

$$|u_{n+1}(Q,\tau)-u_n(Q,\tau)| \le \\ \le \left|\int\limits_0^{\tau}\int\limits_{S_2} R(P,Q;t,\tau)|u_n(P,t)-u_{n-1}(P,\tau)|ds_P dt\right|, \tag{25}$$

where $n(P,\tau)$ $R(P,Q;t-\tau)=\kappa[u_n^3(Q,\tau)+u_n^2(Q,\tau)u_{n-1}(Q,\tau)+u_n(Q;\tau)u_{n-1}^2(Q,\tau)+u_{n-1}^3(Q,\tau)]G(P,Q;t-\tau)$ are positive square integrable functions

$$\int\limits_0^T d\tau\int\limits_{S_2} n^2(Q;\tau)ds_Q \le N^2, \quad N^2=\text{const}, \tag{26}$$

$$\int\limits_0^T d\tau\int\limits_0^{\tau} dt\int\limits_{S_2} ds_P\int\limits_{S_2} R^2(P,Q;t,\tau)ds_Q = \\ = \int\limits_0^T d\tau\int\limits_{S_2} A^2(Q,\tau)ds_Q \le A^2, \quad A^2=\text{const}. \tag{27}$$

From obtained inequalities (24)-(27) and Cauchi-Bunyakovsky inequality we get

$$[u_{n+1}(Q,\tau)-u_0(Q,\tau)]^2 \le \\ \le A^2\int\limits_0^{\tau}\int\limits_{S_2}[u_n(P,t)-u_{n-1}(P,t)]^2 ds_P dt.$$

The sequential substitutions lead to inequality

$$[u_{n+1}(Q,\tau)-u_n(Q,\tau)]^2 \le N^2A^2(Q,\tau)F_{n-1}(\tau), \tag{28}$$

where

$$F_n(\tau)=\int\limits_0^{\tau} dt\int\limits_{S_2} A^2(P,t)F_{n-1}(t)ds_P, \quad n=2,3,...,$$

$$F_1(\tau)=\int\limits_0^{\tau} dt\int\limits_{S_2} A^2(P,t)ds_P \le A^2.$$

Based on method of mathematical induction we can show that

$$F_n(\tau)=\frac{F_1^n(\tau)}{n!}, \quad n=1,2,..., \tag{29}$$

then from (28) and (29) we get inequality

$$|u_{n+1}(Q,\tau)-u_n(Q,\tau)| \le NA(Q,\tau)\frac{A^{n-1}}{\sqrt{(n-1)!}}, \quad n=1,2,..., \tag{30}$$

which provide absolute convergence of functional series

$$U(Q,\tau) = \\ u_1(Q,\tau) + [u_2(Q,\tau) - u_1(Q,\tau)] + [u_3(Q,\tau) - u_2(Q,\tau)] + ..., \tag{31}$$

which n-th partial sum is equal to $u_n(Q,\tau)$.

Indeed, series (31) according estimation (30) can be majorized by convergent series beginning from the second term

$$NA(Q,\tau)\sum_{n=1}^{\infty}\frac{A^{n-1}}{\sqrt{(n-1)!}}.$$

Therefore,

$$\lim_{n\to\infty} u_n(Q,\tau) = U(Q,\tau).$$

We can show that limit function gives solution of integral equation (16). Indeed, assuming

$$U(Q,\tau) = u_n(Q,\tau) + R_n(Q,\tau), \tag{32}$$

we transform (22) to a form

$$U(Q,\tau) - u_l(Q,\tau) + \\ +\kappa\int_0^{\tau}\int_{S_2} G(P,Q;t-\tau)U^4(P,t)ds_P dt = R_n(Q,\tau) - \\ -\kappa\int_0^{\tau}\int_{S_2} G(P,Q;t-\tau)[u_{n-1}^4(P,t) - U^4(P,t)]ds_P dt. \tag{33}$$

By majorizing right side of (33) with accounting obtained estimations and Caochi-Bunyakovsky inequality we get

$$\int_0^{T}\int_{S_2}\{U(Q,\tau) - u_l(Q,\tau) + \\ +\kappa\int_0^{\tau}\int_{S_2} G(P,Q;t-\tau)U^4(P,t)ds_P dt\}ds_Q d\tau \leq \\ \leq 2\int_0^{T}\int_{S_2}[R_n^2(Q,\tau) + A^2 R_{n-1}^2(Q,\tau)]ds_Q d\tau. \tag{34}$$

With proceeding limit transition for $n \to \infty$, we get that integral from left side of (34) is equal zero, as

$$|R_n(Q,\tau)| \leq NA(Q,\tau)\sum_{m=n+1}^{\infty}\frac{A^m}{\sqrt{m!}}.$$

Therefore, limit function $U(Q,\tau)$ satisfy integral equation (16).

Let's show that obtained solution $U(Q,\tau) = u(Q,\tau)$ is unique. Indeed for difference of two solutions $U(Q,\tau) = u^*(Q,\tau)$ we have

$$[u(Q,\tau) - u^*(Q,\tau)]^2 = \{\kappa\iint_{0 S_2}^{\tau} G(P,Q;t-\tau)[u^{*4}(P,t) - u^4(P,t)]ds_P dt\}^2 \leq$$

$$\leq \int_0^\tau \int_{S_2} R^2(P,Q;t,\tau) ds_P dt \int_0^\tau \int_{S_2} [u(P,t) - u^*(P,t)]^2 ds_P dt \leq$$

$$\leq A^2(Q,\tau) \int_0^\tau \int_{S_2} [u(P,t) - u^*(P,t)]^2 ds_P dt = k^2 A^2(Q,\tau).$$

Let's fulfill sequential substitutions in last inequality and account (29). We get

$$\int_0^\tau \int_{S_2} [u(P,t) - u^*(P,t)]^2 ds_P dt \leq \frac{k^2}{n!} \int_0^\tau \int_{S_2} A^2(P,t) ds_P dt \leq \frac{(kA^n)^2}{n!}.$$

Hence for $n \to \infty$ we get $u^*(Q,\tau) = u(Q,\tau)$, that accomplish the proof of Theorem 2.

References

1. *Threcomi F.* Integral equations. - Moscow: Inostr. Liter., 1960. – 299p.
2. *Berezovsky A.A.* Nonlinear boundary problems of heat radiating body. — Kiev: Naukova Dumka, 1968. – 166p.

Institute of Mathematics,
Ukrainian National Academy of Sciences,
Tereshchenkivska, 3, Kyiv, 252602 Ukraine.