Some remarks on a class of elliptic equations with degenerate coercivity

© Lucio Boccardo, Haïm Brezis

1. Some elliptic equations with degenerate coercivity.

In the paper [3], existence and regularity results of the following elliptic problem are studied:

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, and $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (which is measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta\,, \tag{0.2}$$

for some real number θ such that

$$0 \le \theta \le 1\,, \tag{0.3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α and β are positive constants. The datum f belongs to $L^m(\Omega)$, for some $m \ge 1$.

The main difficulty in dealing with problem (0.1) is the fact that, because of assumption (0.2), the differential operator $A(u) = -\mathrm{div}(a(x,u)\nabla u)$, even if well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, is not coercive on $H_0^1(\Omega)$ (when u is, $\frac{1}{(1+|u|)^\theta}$ goes to zero: for an explicit example see [6]).

This implies that the classical methods used in order to prove the existence of a solution for problem (0.1) cannot be applied even if the datum f is regular.

In this note, two new and short proofs of the existence theorems will be presented. The shortness depends on the use of previous results of [7] and [5]; by contrast the original ones of [3] are selfcontained.

We will recall here the existence and regularity results proved in [3], then we will give new proofs of the first two theorems.

The first result concerns the existence of bounded solutions, and coincides with the classical boundedness results for uniformly elliptic operators (see [7]). The main tool of the proof will be an $L^\infty(\Omega)$ *a priori* estimate, which then implies the $H_0^1(\Omega)$ estimate, since if u is bounded then the operator A is uniformly elliptic.

Theorem 0.1 *Let f be a function in $L^m(\Omega)$, with $m > \frac{N}{2}$. Assume (0.2) and (0.3). Then there exists a weak solution of (0.1) u in $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

The next result deals with data f which give unbounded solutions in $H_0^1(\Omega)$.

Theorem 0.2 *Let $0 < \theta < 1$ and f be a function in $L^m(\Omega)$, with m such that*

$$\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}\,. \tag{0.4}$$

Assume (0.2) and $0 < \theta < 1$. Then there exists a function u in $H_0^1(\Omega) \cap L^{m^{**}(1-\theta)}(\Omega)$, which is weak solution of (0.1), where $m^{**} = (m^*)^* = \frac{mN}{N-2m}$.

Remark that, since $\frac{2N}{N+2-\theta(N-2)} \leq m$ and $0 < \theta < 1$, then $m \geq \frac{2N}{N+2}$, so that f belongs to the dual of $H_0^1(\Omega)$.

We recall that Example 1.5 of [3] shows that the result of Theorem 0.2 is sharp.

Remark that if $\theta = 0$, the result of the preceding theorems coincides with the classical regularity results for uniformly elliptic equations (see [7] and [5]).

We refer to [6] for a uniqueness result for (0.1).

If we decrease the summability of f, we find solutions which do not in general belong any more to $H_0^1(\Omega)$, even if our assumptions on f ($f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$) implies that $f \in H^{-1}(\Omega)$.

We recall now other results of [3]

Theorem 0.3 *Let $0 < \theta < 1$ and f be a function in $L^m(\Omega)$, with m such that*

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}\,. \tag{0.5}$$

Then there exists a function u in $W_0^{1,q}(\Omega)$, with

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)} < 2\,, \tag{0.6}$$

which solves (0.1) in the sense of distributions, that is,

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla \varphi = \int_\Omega f\,\varphi\,, \qquad \forall \varphi \in C_0^\infty(\Omega)\,. \tag{0.7}$$

Moreover, the truncation $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$, where

$$T_k(s) = \max\{-k, \min\{k, s\}\}\,. \tag{0.8}$$

Up to now, we have obtained solutions belonging to some Sobolev space. If we weaken the summability hypotheses on f, then the gradient of u (and even u itself) may no longer be in $L^1(\Omega)$. However, it is possible to give a meaning to solution for problem (0.1), using the concept of *entropy solutions* which has been introduced in [2].

Definition 0.4 Let f be a function in $L^1(\Omega)$. A measurable function u is an *entropy solution* of (0.1) if $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$ and if

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla T_k(u-\varphi) \leq \int_\Omega f\,T_k(u-\varphi)\,, \tag{0.9}$$

for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We observe that every term in (0.9) is meaningful. This is clear for the right hand side, while for the left hand side we have

$$\int_\Omega a(x,u)\,\nabla u\cdot\nabla T_k(u-\varphi)=\int_\Omega a(x,u)\,\nabla T_M(u)\cdot\nabla T_k(u-\varphi)\,,$$

where $M=k+\|\varphi\|_{L^\infty(\Omega)}$. Then if f is a function in $L^m(\Omega)$, with

$$1\le m\le\max\left\{\frac{N}{N+1-\theta(N-1)},1\right\}, \tag{0.10}$$

we refer to [3] for the existence of an entropy solution u of (0.1).

Remark that if $0\le\theta<\frac{1}{N-1}$, then (0.10) becomes $m=1$ and $q=\frac{N(1-\theta)}{N-1-\theta}$ which is greater than 1. If in particular $\theta=0$, this is the same result obtained in [4] for elliptic equations with $L^1(\Omega)$ (or measure) data.

If $\frac{1}{N-1}\le\theta<1$, then the upper bound on m is $\frac{N}{N+1-\theta(N-1)}$, which is the lower bound on m given by Theorem 0.3.

2. Proofs.

The proofs of the existence results will be obtained by approximation.

Let f be a function in $L^m(\Omega)$, with m as in the statements of Theorems 0.1, 0.2 and 0.3. Let $\{f_n\}$ be a sequence of functions such that

$$f_n\in L^{\frac{2N}{N+2}}(\Omega)\,,\qquad f_n\to f\quad\text{strongly in } L^m(\Omega), \tag{0.11}$$

and such that

$$\|f_n\|_{L^m(\Omega)}\le\|f\|_{L^m(\Omega)}\,,\qquad\forall n\in\mathbb{N}\,. \tag{0.12}$$

For instance, $f_n=T_n(f)$.

Let us define the following sequence of problems:

$$\begin{cases}-\mathrm{div}(a(x,T_n(u_n))\nabla u_n)=f_n & \text{in }\Omega,\\ u_n=0 & \text{on }\partial\Omega.\end{cases} \tag{0.13}$$

The existence of weak solutions u_n in $H_0^1(\Omega)$ of the Dirichlet problem (0.13) is classical, since the differential operator in (0.13) is uniformly elliptic.

Lemma 0.5 *Assume the same hypotheses of Theorem 1.1. Let f be in $L^m(\Omega)$ and let u_n be a solution of (0.13) with $f_n=f$ for every $n\in\mathbb{N}$. Then the norms of u_n in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$ are bounded by a constant which depends on θ, m, N, α, $|\Omega|$ and on the norm of f in $L^m(\Omega)$.*

Proof. Let us start with the estimate in $L^\infty(\Omega)$. Define, for s in $\mathbb{R}$ and for $k>0$,

$$G_k(s)=(|s|-k)_+\,\mathrm{sgn}(s)=s-T_k(s)\,,$$

and

$$H(s)=\int_0^s\frac{1}{(1+|t|)^\theta}\,dt.$$

For $k > 0$, if we take $G_k(H(u_n))$ as test function in (0.13) and use assumption (0.2), we obtain

$$\alpha \int_{\{x\in\Omega:|H(u_n(x))|>k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\theta}} \le \int_{\{x\in\Omega:|H(u_n(x))|>k\}} f\, G_k(H(u_n)).$$

That is

$$\alpha \int_{A_k} |\nabla(H(u_n))|^2 \le \int_{A_k} f\, G_k(H(u_n)) \tag{0.14}$$

where we have set

$$A_k = \{x \in \Omega \,:\, |H(u_n(x))| > k\}\,.$$

The inequality (0.14) is exactly the starting point of Stampacchia's L^∞-regularity proof (see [7]), so that there exists a constant c_1 such that

$$\|H(u_n)\|_{L^\infty(\Omega)} \le c_1. \tag{0.15}$$

The properties of the functions h and H ($\lim\limits_{s\to\infty} H(s) = +\infty$) yield a bound for u_n in $L^\infty(\Omega)$ from (0.15):

$$\|u_n\|_{L^\infty(\Omega)} \le c_2.$$

The estimate in $H_0^1(\Omega)$ is now very easy. Taking u_n as test function in (0.13), one obtains

$$\frac{\alpha}{(1+c_2)^\theta} \int_\Omega |\nabla u_n|^2 \le \int_\Omega f\, u_n\,,$$

and the right hand side is bounded since f belongs, at least, to $L^1(\Omega)$. ■

The next result will be used in the proof of Theorem 0.2.

Lemma 0.6 *Assume the same hypotheses as in Theorem 1.2. Let f belong to $L^\infty(\Omega)$, and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (0.1) (which exists by Theorem 1.1). Then the norms of u in $L^{m^{**}(1-\theta)}(\Omega)$ and in $H_0^1(\Omega)$ are bounded by constants depending only on θ, m, N, α, $|\Omega|$ and the norm of f in $L^m(\Omega)$.*

Proof. Multiplying (0.1) by $\gamma(u)$, where

$$\gamma(t) = ((1+|t|)^p - 1)\mathrm{sign}(t), \quad p = \frac{(1-\theta)N(m-1)}{N-2m},$$

integrating on Ω and using the standard Sobolev imbedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, yields

$$\|u\|_{L^{(p+1-\theta)2^*/2}(\Omega)} \le c_3, \tag{0.16}$$

and

$$\int_\Omega |\nabla u|^2(1+|u|)^{p-1-\theta} \le c_4. \tag{0.17}$$

It is convenient to observe that $pm' = (p+1-\theta)2^*/2$. So far, we have not used (0.4). Note that clearly

$$(p+1-\theta)2^*/2 = m^{**}(1-\theta)$$

and that (0.4) is equivalent to $p-1-\theta \ge 0$. Thus, if (0.4) holds, then (0.16) implies a bound for ∇u in $L^2(\Omega)$. Here the constants c_3 and c_4 depend only on θ, m, N, α, $|\Omega|$ and the norm of f in $L^m(\Omega)$.

■

Proof of Theorems 0.1 and 0.2

The estimates for u_n in $H_0^1(\Omega)$ imply that there exist a subsequence $\{u_{n_j}\}$ and a function $u \in H_0^1(\Omega)$, such that u_{n_j} converges weakly in $H_0^1(\Omega)$ to u. The coefficient $a(x, u_{n_j})$ converges to $a(x, u)$ in any $L^p(\Omega)$. Thus it is possible to pass to the limit in (0.13) and to obtain the existence of a weak solution u. ■

Remark 0.7 We point out that Lemma 0.5 can be proved under the slightly more general assumption

$$h(s) \leq a(x, s) \leq \beta, \tag{0.18}$$

where the real function $h(s)$ is continuous, decreasing, strictly positive and such that its primitive

$$H(s) = \int_0^s h(t)\, dt \tag{0.19}$$

is unbounded.

Thus it is possible, for instance, to study also problems where $h(s) = \frac{1}{(e+|s|)\ln(e+|s|)}$.

3. Lower order terms.

The presence of lower order terms in the Dirichlet problem (0.1) can change the existence results. For instance, consider the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(x, u)\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.20}$$

where $a(x, s)$ still satisfies the following inequality

$$\frac{\alpha}{(1 + |s|)^\theta} \leq a(x, s) \leq \beta\,, \tag{0.21}$$

for some real number $\theta > 0$ (for almost every $x \in \Omega$, for every $s \in I\!R$, α, $\beta > 0$) and f belongs to $L^m(\Omega)$, for some $m \geq 1$.

Let $\{f_n\}$ be the sequence of functions

$$f_n = T_n(f) \tag{0.22}$$

Define the following sequence of problems:

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n))\nabla u_n) + u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.23}$$

It is classical that

Lemma 0.8

$$\|u_n\|_{L^m(\Omega)} \leq \|f_n\|_{L^m(\Omega)}.$$

■

Now we can prove that

Lemma 0.9 *Assume*

$$m \geq \theta + 2. \tag{0.24}$$

Then the sequence $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$.

Proof. The use of

$$[(1 + |u_n|)^{1+\theta} - 1]\text{sign}(u_n)]$$

as test function in (0.23) implies that

$$\int_\Omega |\nabla u_n|^2 \leq c_5\{1 + \|f_n\|_{L^m(\Omega)}\}\||u_n|^{(1+\theta)}\|_{L^{m'}(\Omega)}.$$

Remark that $(1 + \theta)m' \leq m$ if and only if $m \geq 2 + \theta$.

■

Then, if f belongs to $L^m(\Omega)$ and if (0.24) holds, the existence of solutions follows as in the proofs of Theorems 1.1 and 1.2.

References

[1] A. Alvino, V. Ferone, G. Trombetti, *A priori estimates for a class of non uniformly elliptic equations*, to appear on Atti Sem. Mat. Univ. Modena.

[2] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, *An* L^1 *theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 22 (1995), 240–273.

[3] L. Boccardo, A. Dall'Aglio, L. Orsina, *Existence and regularity results for some elliptic equations with degenerate coercivity* , to appear on Atti Sem. Mat. Univ. Modena.

[4] L. Boccardo, T. Gallouët, *Nonlinear elliptic equations with right-hand side measures*, Comm. Partial Differential Equations 17 (1992), 641–655.

[5] L. Boccardo, D. Giachetti, L^s*-regularity of solutions of some nonlinear elliptic problems*, preprint.

[6] A. Porretta, *Uniqueness and homogenization for a class of non coercive operators in divergence form*, to appear on Atti Sem. Mat. Univ. Modena.

[7] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)*, 15 (1965), 189–258.

Lucio Boccardo
Dipartimento di Matematica, Universita' di Roma 1
Piazza A. Moro 2, 00185 Roma
Tel: +39.06.49.91.32.02; fax: +39.06.44.701.007
E-mail: boccardo@gprme.sci.uniromal.il

Analyse Numerique
Universite P.et M.Curie
4, pl. Jussieu, 75252 Paris
Cedex 05, France
E-mail: brezis@ccr.jussie.fr