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Weighted sharp inequality for vector-valued multilinear commutator of strongly singular integral operator

Hongwei Zhang, Haitao Cai

(*Presented by M. M. Malamud*)

Abstract. In this paper, a sharp inequality for the vector-valued multilinear commutator of strongly singular integral operator are obtained. By using this inequality, we prove the weighted L^p-norm inequality for the multilinear commutator.

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1. Introduction and Theorems

As the development of Calderón–Zygmund singular integral operators, the commutators of the singular integral operators have been well studied (see [7–11]). In this paper, we study the vector-valued multilinear commutator of strongly singular integral operator which is defined as following.

Let $0 < b < 1$ and $\theta(\xi)$ be a smooth radial cut-off function on $\mathbb{R}^n$ such that $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. The strongly singular integral operator is a multiplier operator which is defined by

$$(T(f))\hat{}(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^{nb/2}}\hat{f}(\xi).$$

The kernel K for T is very singular. Roughly speaking, it looks like $K(x) = \vartheta(x)e^{i|x|^{-b'}}/|x|^n$, where $b' = b/(1-b)$ and supp $\vartheta \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$. In fact, we know that

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$$T(f)(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

and for $|x| \geq 2|y|$ (see [1])

$$|K(x-y) - K(x)| \leq C|y||x|^{-n-b'-1}.$$

Let $b_j (j = 1, \ldots, m)$ be the fixed locally integrable functions on $\mathbb{R}^n$. For $1 < s < \infty$, the vector-valued multilinear commutator related to T is defined by

$$|T_{\tilde{b}}(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_{\tilde{b}}(f_i)(x)|^s \right)^{1/s},$$

where

$$T_{\tilde{b}}(f_i)(x) = p.v. \int_{\mathbb{R}^n} \left(\prod_{j=1}^{m} (b_j(x) - b_j(y)) \right) K(x-y) f_i(y)\,dy.$$

We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^s \right)^{1/s} \quad \text{and} \quad |f(x)|_s = \left(\sum_{i=1}^{\infty} |f_i(x)|^s \right)^{1/s}.$$

The strongly singular integral operator has been studied by several authors (see [1, 4, 5, 13]). In [2], the weighted norm inequality for the commutator of strongly singular integral operator is obtained by using a sharp estimate. Note that when $b_1 = \cdots = b_m$, $|T_{\tilde{b}}|_s$ is just the m order vector-valued commutator (see, for example, [6, 10] and [11]). In [11], Perez and Trujillo–Gonzalez prove a sharp estimate for the vector-valued commutator of Calderon–Zygmund singular integral operator. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear commutator of strongly singular integral operator. As an application, we obtain the weighted L^p-norm inequality for the multilinear commutator.

First, let us introduce some notations. Throughout this paper, $Q = Q(x, r)$ will denote a cube of R^n centered at x and sidelength r with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int\limits_{Q}|f(y)-f_Q|\,dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)\,dx$. It is well-known that (see, for example, [3] and [12])

$$f^{\#}(x)\cong\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int\limits_{Q}|f(y)-c|\,dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO}=\|f^{\#}\|_{L^\infty}$. Given the functions $b_j (j=1,\dots,m)$ and $0<q<\infty$, denote that

$$C_{\tilde{b}}^{q}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int\limits_{Q}\prod_{j=1}^{m}|b_j(x)-b_j(y)|^q|f(y)|\,dy.$$

We write $C_{\tilde{b}}^{q}(f)=C_{\tilde{b}}(f)$ if $q=1$. Let M be the Hardy–Littlewood maximal operator defined by

$$M(f)(x)=\sup_{Q\ni x}|Q|^{-1}\int\limits_{Q}|f(y)|\,dy.$$

We write $M_p(f)=(M(f^p))^{1/p}$ for $0<p<\infty$. Given a positive integer m and $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\dots,\sigma(j)\}$ of $\{1,\dots,m\}$ of j different elements. For $\sigma\in C_j^m$, denote that $\sigma^c=\{1,\dots,m\}\setminus\sigma$; For $\tilde{b}=(b_1,\dots,b_m)$ and $\sigma=\{\sigma(1),\dots,\sigma(j)\}\in C_j^m$, denote $\tilde{b}_\sigma=(b_{\sigma(1)},\dots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\dots b_{\sigma(j)}$ $\|\tilde{b}\|_{BMO}=\|b_1\|_{BMO}\dots\|b_m\|_{BMO}$ and $\|\tilde{b}_\sigma\|_{BMO}=\|b_{\sigma(1)}\|_{BMO}\cdots\times\|b_{\sigma(j)}\|_{BMO}$. We denote the Muckenhoupt weights by A_p for $1\le p<\infty$ (see [3]), that is

$$A_p=\left\{w:\sup_{Q}\left(\frac{1}{|Q|}\int\limits_{Q}w(x)\,dx\right)\left(\frac{1}{|Q|}\int\limits_{Q}w(x)^{-1/(p-1)}\,dx\right)^{p-1}<\infty\right\},\quad 1<p<\infty,$$

and

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

We shall prove the following theorems.

Theorem 1.1. *Let $1 < s < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \ldots, m$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < \infty$ and $\tilde{x} \in \mathbb{R}^n$,*

$$(|T_{\tilde{b}}(f)|_s)^{\#}(\tilde{x}) \le C\Big(\|\tilde{b}\|_{BMO} M_r(|f|_s)(\tilde{x}) + M((C_{\tilde{b}}^s(|f|_s^r))^{1/r})(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \|\tilde{b}_\sigma\|_{BMO} M_r(|T_{\tilde{b}_{\sigma^c}}(f)|_s)(\tilde{x})\Big).$$

Theorem 1.2. *Let $1 < s < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \ldots, m$ and $w \in A_p$, $1 < p < \infty$. Then $|T_{\tilde{b}}|_s$ is bounded on $L^p(w)$, that is*

$$\| \, |T_{\tilde{b}}(f)|_s \|_{L^p(w)} \le C \| \, |f|_s \|_{L^p(w)}.$$

2. Proof of the main results

To prove the theorems, we need the following lemmas.

Lemma 2.1 ([1]). *Let T be the strongly singular integral operator and $1 < s < \infty$. Then $|T|_s$ is bounded on $L^p(w)$ for $w \in A_p$ and $1 < p < \infty$.*

Lemma 2.2 ([1]). *Let $0 < b < 1$, $b' = b/(1-b)$, $1 < p < \infty$, $1/p + 1/p' = 1$, $(2 + b')/p \le 1$, $1 < s < \infty$ and*

$$\tilde{K}(x) = \frac{e^{i|x|^{-b'}}}{|x|^{n(b'+2)/p}}.$$

Then

$$\|\tilde{K} * |f|_s\|_{L^p} \le C \| \, |f|_s \|_{L^{p'}}.$$

Lemma 2.3 ([3]). *Suppose that $1 < s < \infty$, $1 \le r < p < \infty$ and $w \in A_p$. Then*

$$\|M_r(|f|_s)\|_{L^p(w)} \le C \| \, |f|_s \|_{L^p(w)}.$$

Lemma 2.4 ([2]). *Suppose that $b_j \in BMO(R^n) (j = 1, \ldots, m)$, $1 < s < \infty$, $1 \le q < \infty$, $1 < p < \infty$ and $w \in A_p$. Then*

$$\|C_{\tilde{b}}^q(|f|_s)\|_{L^p(w)} \le C \| \, |f|_s \|_{L^p(w)}.$$

The Proofs of Lemma 2.1, 2.3 and 2.4 is similar to the proofs in [1–3], we omit the detail.

Proof of Theorem 1.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \big|\,|T_{\tilde{b}}(f)(x)|_s - C_0\big|\,dx \le C\Bigg(\|\tilde{b}\|_{BMO} M_r(|f|_s)(\tilde{x}) + M((C_{\tilde{b}}^r(|f|_s^r))^{1/r})(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\|\tilde{b}_\sigma\|_{BMO} M_r(|T_{\tilde{b}_{\sigma^c}}(f)|_s)(\tilde{x})\Bigg).$$

Fix a cube $Q = Q(x_0, d)$ such that $\tilde{x} \in Q$. Let d_0 be a positive number satisfying $4d_0 = d_0^{1/(1+b')}$ and $\tilde{Q} = Q(x_0, d^{1/(1+b')})$. Let us first consider the case $m = 1$. In this time, we will prove the following inequality

$$\frac{1}{|Q|}\int_Q \big|\,|T_{\tilde{b}}(f)(x)|_s - C_0\big|\,dx \le C\big(M_r(|f|_s)(\tilde{x}) + M((C_{\tilde{b}}^r(|f|_s^r))^{1/r})(\tilde{x}) + M_r(|T(f)|_s)(\tilde{x})\big).$$

Consider the following two cases:

Case 1. $d < d_0$. For each $i \in N$, we split $f_i = f_i^1 + f_i^2 + f_i^3$, where $f_i^1 = f_i\chi_{4Q}$, $f_i^2 = f_i\chi_{\tilde{Q}\setminus 4Q}$ and $f_i^3 = f_i\chi_{R^n\setminus\tilde{Q}}$ and we define $f^{(j)} = \{f_i^j\}$ for $j = 1, 2, 3$. By Minkowski' inequality and taking $C_0 = |T_{\tilde{b}}(f^{(3)})(x_0)|_s$, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \big|\,|T_{\tilde{b}}(f)(x)|_s - |T_{\tilde{b}}(f^{(3)})(x_0)|_s\big|\,dx
&\le \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}|T_{\tilde{b}}(f_i)(x) - T_{\tilde{b}}(f_i^3)(x_0)|^s\right)^{1/s} dx\\
&\le \frac{1}{|Q|}\int_Q |b(x) - b_{2Q}|\,|T(f)(x)|_s\,dx\\
&\quad + \frac{1}{|Q|}\int_Q |T((b - b_{2Q})f^{(1)})(x)|_s\,dx
\end{aligned}$$

$$+\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f^{(2)})(x)|_s\,dx$$

$$+\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f^{(3)})(x)-T((b-b_{2Q})f^{(3)})(x_0)|_s\,dx$$

$$=I_1+I_2+I_3+I_4.$$

Let us now estimate I_1, I_2, I_3 and I_4, respectively. By applying Hölder's inequality and Lemma 2.1, we get

$$I_1\le C\left(\frac{1}{|Q|}\int_Q |T(f)(x)|_s^r\,dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{r'}\,dx\right)^{1/r'}$$
$$\le C\|b\|_{BMO}M_r(|T(f)|_s)(\tilde{x}).$$

Now we proceed to estimate I_2. If $1<p<r$, from Lemma 2.1, we obtain

$$I_2\le C\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T((b-b_{2Q})f^{(1)}(x))(x)|_s^p\,dx\right)^{1/p}$$
$$\le\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|b(x)-b_{2Q}|^p|f^{(1)}(x)|_s^p\,dx\right)^{1/p}$$
$$\le C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|_s^r\,dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{pr/(r-p)}\,dx\right)^{(r-p)/pr}$$
$$\le C\|b\|_{BMO}M_r(|f|_s)(\tilde{x}).$$

To estimate I_3, we follow Chanillo's argument (see [1]). Then we choose $r>1$ such that $(2+b')/r'<1$ to get

$$T((b-b_{2Q})f_i^2(x)$$
$$:=\int_{\mathbb{R}^n}\frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}\Big(\frac{1}{|x-y|^{n(1-(2+b')/r')}}$$
$$-\frac{1}{|x_0-y|^{n(1-(2+b)/r')}}\Big)(b(y)-b_{2Q})f_i^2(y)\,dy$$

$$+\int_{\mathbb{R}^n}\frac{\vartheta(x-y)e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}\cdot\frac{1}{|x_0-y|^{n(1-(2+b')/r')}}(b(y)-b_{2Q})f_i^2(y)\,dy$$

$$= I_3^{(1)}(x) + I_3^{(2)}(x).$$

Note that $|b_{2Q} - b_{2^{k+1}Q}| \leq k\|b\|_{BMO}$, then, by Minkowski' inequality,

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty} \left|I_3^{(1)}(x)\right|^s\right)^{1/s} \\
&\qquad \leq C \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(y) - b_{2Q}\|f(y)|_s \, dy \\
&\qquad \leq C \sum_{k=1}^{\infty} 2^{-k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(x)|_s^r \, dx\right)^{1/r} \\
&\qquad\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(x) - b_{2Q}|^{r'} \, dx\right)^{1/r'} \\
&\qquad \leq C\|b\|_{BMO} \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(x)|_s^r \, dx\right)^{1/r} \\
&\qquad \leq C\|b\|_{BMO} M_r(|f|_s)(\tilde{x}).
\end{aligned}$$

Taking k_0 such that $2^{k_0}d < d^{1/(1+b')} \leq 2^{k_0+1}d$, by Lemma 2.2 and Minkowski' inequality, we get

$$\begin{aligned}
&\frac{1}{Q} \int_Q \left(\sum_{i=1}^{\infty} \left|I_3^{(2)}(x)\right|^s\right)^{1/s} dx \\
&\qquad \leq C|Q|^{-1/r'} \left(\int_{\mathbb{R}^n} \frac{|b(y) - b_{2Q}|^r |f^{(2)}(y)|_s^r}{|x_0 - y|^{nr(1-(2+b')/r')}} \, dy\right)^{1/r} \\
&\qquad \leq C|Q|^{-1/r'} \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \frac{1}{|2^{k+1}Q|}\right. \\
&\qquad\quad \left. \times \int_{2^{k+1}Q} |f(y)|_s^r |b(y) - b_{2Q}|^r \, dy\right)^{1/r} \\
&\qquad \leq C|Q|^{-1/r'} \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \left[\frac{1}{|2Q|} \int_{2Q} \left(\frac{1}{|2^{k+1}Q|}\right.\right.\right.
\end{aligned}$$

$$\times \int_{2^{k+1}Q} |f(y)|_s^r |b(y)-b(z)|^r \, dy\Bigg)^{1/r} dz\Bigg]^r\Bigg)^{1/r}$$

$$\leq CM((C_{\tilde{b}}^r(|f|_s^r))^{1/r})(\tilde{x}).$$

Thus

$$I_3 \leq C\big(\|b\|_{BMO} M_r(|f|_s)(\tilde{x}) + M((C_b^r(|f|_s^r))^{1/r})(\tilde{x})\big).$$

For I_4, note that $|x-y| \approx |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition on K,

$$\begin{aligned}
I_4 &\leq C\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n} \frac{|x-x_0|}{|x_0-y|^{n+b'+1}} |f^{(3)}(y)|_s |b(y)-b_{2Q}| \, dy \, dx \\
&\leq C\frac{1}{|Q|}\int_Q \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+b'+1}} |f(y)|_s |b(y)-b_{2Q}| \, dy \, dx \\
&\leq C\sum_{k=0}^{\infty} 2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |f(y)|_s |b(y)-b_{2Q}| \, dy \\
&\leq C\sum_{k=1}^{\infty} 2^{-k}\frac{1}{|2Q|}\int_{2Q}\Bigg(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |f(y)|_s |b(y)-b(z)| \, dy\Bigg) dz \\
&\leq CM(C_{\tilde{b}}(|f|_s))(\tilde{x}).
\end{aligned}$$

Case 2. $d \geq d_0$. We do not subtract the constant C_0. Let $l = 4d_0^{-1}$ and $f_i^{(4)} = f_i\chi_{lQ}$, by the location of the support of K, we have $T_{\tilde{b}}(f_i) = T_{\tilde{b}}(f_i^{(4)})$, thus

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x)|_s \, dx \\
&\leq \frac{1}{|Q|}\int_Q |b(x)-b_{lQ}|\,|T(f^{(4)})(x)|_s \, dx \\
&\quad + \frac{1}{|Q|}\int_Q |T((b-b_{lQ})f^{(4)})(x)|_s \, dx = J_1 + J_2.
\end{aligned}$$

Similar to the proof of I_1 and I_2 for case 1, we get

$$J_1 \leq C\Bigg(\frac{1}{|Q|}\int_Q |T(f^{(4)})(x)|_s^r \, dx\Bigg)^{1/r}\Bigg(\frac{1}{|lQ|}\int_Q |b(x)-b_{lQ}|^{r'} \, dx\Bigg)^{1/r'}$$

$$\leq C\left(\frac{1}{|lQ|}\int\limits_{lQ}|f(x)|_s^r\,dx\right)^{1/r}\left(\frac{1}{|lQ|}\int\limits_{Q}|b(x)-b_{lQ}|^{r'}\,dx\right)^{1/r'}$$
$$\leq C\|b\|_{BMO}M_r(|f|_s)(\tilde{x});$$

$$J_2\leq C\left(\frac{1}{|Q|}\int\limits_{\mathbb{R}^n}|T((b-b_{lQ})f^{(4)})(x)|_s^p\,dx\right)^{1/p}$$
$$\leq C\left(\frac{1}{|Q|}\int\limits_{\mathbb{R}^n}|b(x)-b_{lQ}|^p|f^{(4)}(x)|_s^p\,dx\right)^{1/p}$$
$$\leq C\left(\frac{1}{|lQ|}\int\limits_{lQ}|f(x)|_s^r\,dx\right)^{1/r}\left(\frac{1}{|lQ|}\int\limits_{lQ}|b(x)-b_{lQ}|^{pr/(r-p)}\,dx\right)^{(r-p)/pr}$$
$$\leq C\|b\|_{BMO}M_r(|f|_s)(\tilde{x}),$$

which proves the case 1.

Now we turn to the case $m\geq 2$. Also consider the following two cases:

Case 1. $d<d_0$. Following [10], we write

$$T_{\tilde{b}}(f_i)(x)=\int\limits_{\mathbb{R}^n}\left(\prod_{j=1}^{m}(b_j(x)-b_j(y))\right)K(x-y)f_i(y)\,dy$$
$$=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f_i)(x)$$
$$+(-1)^mT((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_i)(x)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int\limits_{\mathbb{R}^n}(b(y)-b(x))_{\sigma^c}K(x-y)f_i(y)\,dy$$
$$=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f_i)(x)$$
$$+(-1)^mT((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_i)(x)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma T_{\tilde{b}_{\sigma^c}}(f_i)(x),$$

thus, for $f^{(j)}=\{f_i^j\}$ for $j=1,2,3$ with $f_i^1=f_i\chi_{4Q}$, $f_i^2=f_i\chi_{\tilde{Q}\setminus 4Q}$ and $f_i^3=f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\frac{1}{|Q|}\int\limits_Q \left||T_{\tilde{b}}(f)(x)|_s - |T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f^{(3)})(x_0)|_s\right| dx$$

$$\leq \frac{1}{|Q|}\int\limits_Q |(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x)|_s\, dx$$

$$+\frac{1}{|Q|}\int\limits_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} |(b(x)-(b)_{2Q})_\sigma T_{\tilde{b}_{\sigma^c}}(f)(x)|_s\, dx$$

$$+\frac{1}{|Q|}\int\limits_Q |T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f^{(1)})(x)|_s\, dx$$

$$+\frac{1}{|Q|}\int\limits_Q |T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f^{(2)})(x)|_s\, dx$$

$$+\frac{1}{|Q|}\int\limits_Q |T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f^{(3)})(x)$$

$$-T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f^{(3)})(x_0)|_s\, dx$$

$$= L_1+L_2+L_3+L_4+L_5.$$

Similar to the proof of $m=1$, we get, for $1<p_1,\ldots,p_m<\infty$, $1<p<r$, $1<q_1,\ldots,q_m<\infty$, $1/r+1/p_1+\cdots+1/p_m=1$ and $p/r+1/q_1+\cdots+1/q_m=1$,

$$L_1 \leq C\left(\frac{1}{|Q|}\int\limits_Q |T(f)(x)|_s^r\, dx\right)^{1/r}\left(\frac{1}{|Q|}\int\limits_Q |b_1(x)-(b_1)_{2Q}|^{p_1}\, dx\right)^{1/p_1}$$

$$\times\cdots\times\left(\frac{1}{|2Q|}\int\limits_{2Q} |b_m(x)-(b_m)_{2Q}|^{p_m}\, dx\right)^{1/p_m}$$

$$\leq C\prod_{j=1}^m \|b_j\|_{BMO} M_r(|T(f)|_s)(\tilde{x});$$

$$L_2 \leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} \|\tilde{b}_\sigma\|_{BMO} M_r(|T_{\tilde{b}_{\sigma^c}}(f)|_s)(\tilde{x});$$

$$L_3 \leq C\left(\frac{1}{|Q|}\int\limits_{\mathbb{R}^n} |T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f^{(1)})(x)|_s^p\, dx\right)^{1/p}$$

$$\leq C\left(\frac{1}{|Q|}\int\limits_{\mathbb{R}^n}(|b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}|\,|f^{(1)}(x)|)^p\,dx\right)^{1/p}$$

$$\leq C\left(\frac{1}{|2Q|}\int\limits_{2Q}|f(x)|_s^r\,dx\right)^{1/r}\left(\frac{1}{|2Q|}\int\limits_{2Q}|b_1(x)-(b_1)_{2Q}|^{pq_1}\,dx\right)^{1/pq_1}$$

$$\times\cdots\times\left(\frac{1}{|2Q|}\int\limits_{2Q}|b_m(x)-(b_m)_{2Q}|^{pq_m}\,dx\right)^{1/pq_m}$$

$$\leq C\prod_{j=1}^m\|b_j\|_{BMO}M_r(|f|_s)(\tilde{x}).$$

Similarly, for L_4, we get, for $1<p_1,\dots,p_m<\infty$ and $1/r+1/p_1+\cdots+1/p_m=1$,

$$L_4\leq\frac{1}{Q}\int\limits_Q\int\limits_{\mathbb{R}^n}\frac{|\vartheta(x-y)|e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}$$

$$\times\left|\frac{1}{|x-y|^{n(1-(2+b')/r')}}-\frac{1}{|x_0-y|^{n(1-(2+b)/r')}}\right|$$

$$\times\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}|\,|f^{(2)}(y)|_s\,dy\,dx$$

$$+\frac{1}{Q}\int\limits_Q\int\limits_{\mathbb{R}^n}\frac{|\vartheta(x-y)|e^{i|x-y|^{-b'}}}{|x-y|^{n(2+b')/r'}}\cdot\frac{1}{|x_0-y|^{n(1-(2+b')/r')}}$$

$$\times\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}|\,|f^{(2)}(y)|_s\,dy\,dx$$

$$\leq C\sum_{k=1}^\infty 2^{-k}\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}||f(y)|_s\,dy$$

$$+C|Q|^{-1/r'}\left(\int\limits_{\mathbb{R}^n}\frac{\prod_{j=1}^m|b_j(y)-(b_j)_{2Q}|^r|f^{(2)}(y)|_s^r}{|x_0-y|^{nr(1-(2+b')/r')}}\,dy\right)^{1/r}$$

$$\leq C\sum_{k=1}^\infty 2^{-k}\left(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}|f(x)|_s^r\,dx\right)^{1/r}$$

$$\times \prod_{j=1}^{m} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_j(x) - (b_j)_{2Q}|^{p_j'}\, dx \right)^{1/p_j'}$$

$$+ C|Q|^{-1/r'} \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \frac{1}{|2^{k+1}Q|} \right.$$

$$\left. \times \int_{2^{k+1}Q} |f(y)|_s^r \prod_{j=1}^{m} |b_j(y) - (b_j)_{2Q}|^r\, dy \right)^{1/r}$$

$$\le C \prod_{j=1}^{m} \|b_j\|_{BMO} \sum_{k=1}^{\infty} k^m 2^{-k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(x)|_s^r\, dx \right)^{1/r}$$

$$+ C|Q|^{-1/r'} \left(\sum_{k=1}^{k_0} (2^k d)^{n(r-1)(1+b')} \left[\frac{1}{|2Q|} \int_{2Q} \left(\frac{1}{|2^{k+1}Q|} \right. \right. \right.$$

$$\left. \left. \left. \times \int_{2^{k+1}Q} |f(y)|_s^r \prod_{j=1}^{m} |b_j(y) - (b_j)(z)|^r\, dy \right)^{1/r} dz \right]^r \right)^{1/r}$$

$$\le C \left(\prod_{j=1}^{m} \|b_j\|_{BMO} M_r(|f|_s)(\tilde{x}) + M((C_{\tilde{b}}^r(|f|_s^r))^{1/r})(\tilde{x}) \right).$$

For L_5, we get

$$L_5 \le C \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \frac{|x - x_0|}{|x_0 - y|^{n+b'+1}} |f^{(3)}(y)|_s \prod_{j=1}^{m} |b_j(y) - (b_j)_{2Q}|\, dy\, dx$$

$$\le C \frac{1}{|Q|} \int_Q \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+b'+1}} |f(y)|_s \prod_{j=1}^{m} |b_j(y) - (b_j)_{2Q}|\, dy$$

$$\le C \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^k \tilde{Q}|} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |f(y)|_s \prod_{j=1}^{m} |b_j(y) - (b_j)_{2Q}|\, dy$$

$$\le C \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2Q|} \int_{2Q} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(y)|_s \prod_{j=1}^{m} |b_j(y) - b_j(z)|\, dy \right) dz$$

$$\le CM(C_{\tilde{b}}(|f|_s))(\tilde{x}).$$

Similarly, for case 2 $d \geq d_0$, we get

$$\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x)|_s\,dx \leq C\prod_{j=1}^{m}\|b_j\|_{BMO}M_r(|f|_s)(\tilde{x}) + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\tilde{b}_\sigma\|_{BMO}M_r(|T_{\tilde{b}_{\sigma^c}}(f)|_s)(\tilde{x}).$$

These complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. We choose $1 < r < p$ in Theorem 1.1, and by using Lemma 2.3, 2.4 and induction on m, we get

$$\||T_{\tilde{b}}(f)|_s\|_{L^p(w)} \leq C\|(|T_{\tilde{b}}(f)|_s)^{\#}\|_{L^p(w)} \leq C\||f|_s\|_{L^p(w)}.$$

This finishes the proof. □

Remark 2.1. The Theorem 1.1 and 1.2 also hold for $b_j \in Osc_{\exp L^{r_j}}(\mathbb{R}^n)$, where $Osc_{\exp L^{r_j}}(\mathbb{R}^n)$ is defined in [10]. It is obvious that $Osc_{\exp L^r}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and $Osc_{\exp L^r}(\mathbb{R}^n)$ coincides with the $BMO(\mathbb{R}^n)$ space if $r=1$.

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Contact information

Hongwei Zhang, Haitao Cai

School of Mathematical Sciences and Computing Technology
Central South University
Changsha University of Science and Technology
Changsha, P.R. of China
China
E-Mail: zhouxiaosha57@126.com