Український математичний вісник
Том 2 **(2005)**, № 1, 15 – 37

Entropy of the Shift on II_1-representations of the Group $S(\infty)$

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Abstract. We have obtained the explicit formulae for the CNT-entropy of the shift on II_1-representations of the infinite symmetric group $S(\infty)$ in terms of Thoma-parameters.

2000 MSC. 37A35; 37B40; 20C32.

Key words and phrases. CNT-entropy, factor representation, infinite symmetric group.

1. Introduction

Entropy is one of the most important notion in the information theory and the ergodic theory. Initially entropy has appeared in the Claude Shannon's applied works. Next Kolmogorov and Sinai developed the important invariant, namely the entropy for an automorphism of an Abelian $W^\star$-algebra (see [9], [15], [16]). In 1975 the entropy for an automorphism of a non-abelian $W^\star$-algebra with a central state was defined by A. Connes and E. Størmer (see [5]). The final definition was given in the paper of Connes, Narnhofer and Tirring in 1987 (see [4]). This one is usually called by the quantum dynamical entropy or the CNT-entropy.

The CNT-entropy is calculated for many non-commutative dynamical systems of the topological, algebraic or physical origin. We consider in our work the dynamical systems generated by the shift automorphism on the II_1-representations of the infinite symmetric group $S(\infty)$. The group $S(\infty)$ has been often quoted as a typical example of ICC-groups and hence of groups of non-type I. For that reason $S(\infty)$ involves a number of interesting features which not observed in groups of type I. Dynamical systems generated by the non-commutative shift have been

Received 25.02.2004
Supported in part by CRDF grant UM1-2546

ISSN 1810 – 3200.

investigated beginning from the introduction of the notion of the CNT-entropy. Connes and Størmer obtained the explicit formulae for the non-commutative Bernoulli shift (see [5]). In the work of Størmer and Golodets the similar results was obtained for the binary shift on a CAR-algebra (see [7]). The main examples for which the C^*-algebra entropy have been computed, are those of quasifree states of the CAR and CCR-algebras and invariant Bogoliubov (or quasifree) automorphisms (see [2], [6], [11], [12], [14], [17]). In our work [3] the Bogoliubov automorphisms on the II_1-representations of $U(\infty)$ are defined and the explicit formulae for the CNT-entropy are obtained in the case of elementary characters. Using the results of the present work for a low estimation of the CNT-entropy of the shift on the II_1-representations of $U(\infty)$ we obtain the formulae for the Bogoliubov automorphism in the case of a general character (this results will be published in the separate paper).

Denote by $S(2n+1)=S\left(\mathcal{B}_n\right)$ the group of permutations of the set $\mathcal{B}_n = \{-n,\ldots,0,\ldots,n\}$. If $\mathcal{A}$ and $\mathcal{B}$ are two sets and $\mathcal{B}\subset\mathcal{A}$, then we identify $S\left(\mathcal{B}\right)$ with the subgroup $\{g\in S\left(\mathcal{A}\right):\ ga=a\ \forall\, a\in\mathcal{A}\backslash\mathcal{B}\}$ of $S\left(\mathcal{A}\right)$. Let $S(\infty)=\bigcup\limits_n S(2n+1)$. Thoma has obtained the full description of II_1-factor-representations of group $S(\infty)$. Corresponding normalized characters $\chi^{(S)}_{\alpha,\beta}$ are labelled by a pair of sequences of real numbers $\{\alpha_i\}=\alpha$, $\{\beta_i\}=\beta$, $i=1,2,\ldots$, such that $\alpha_i\geq\alpha_{i+1}\geq 0,\ \beta_j\geq\beta_{j+1}\geq 0\ \forall\, i,j\in\mathbb{N}$, $\sum\alpha_i+\sum\beta_j\leq 1$. The value of a character $\chi^{(S)}_{\alpha,\beta}$ on a permutation with a single cycle of length k is equal to

$$\sum_j \alpha_j^k+(-1)^{k-1}\sum_j\beta_j^k \tag{1.1}$$

Its value on a permutation with several disjoint cycles equals to the product of its values on each cycle. As usual, it is assumed that an empty product equals to 1. In particular, the character of the regular representation of the group $S(\infty)$ corresponds to the sequences $\alpha_j\equiv 0$, $\beta_j\equiv 0$.

The bijection $i\in\mathbb{Z}\to i+1\in\mathbb{Z}$ defines naturally an automorphism ϑ_S of the group $S(\infty)$, which extends up to the automorphism ϑ_S^χ of the II_1-factor built by the representation that corresponds to the character χ. We denote by $H_\chi(\theta)$ the CNT-entropy of an automorphism θ of the II_1-factor.

The main result of our work is following

Theorem 1.1. *Let* $\chi=\chi^{(S)}_{\alpha,\beta}$, *let* $\eta(t)=-t\ln t$ *and* $\gamma=1-(\sum\alpha_i+\sum\beta_j)$.

(*i*) *If* $\gamma>0$ *then* $H_\chi\left(\vartheta_S^\chi\right)=\infty$.

(*ii*) *If* $\gamma=0$ *then* $H_\chi\left(\vartheta_S^\chi\right)=\sum\limits_j\eta\left(\alpha_j\right)+\sum\limits_j\eta\left(\beta_j\right)$.

2. The Case $\gamma > 0$

In this section we will consider the case $\gamma > 0$.

Theorem 2.1. *Let* $\chi = \chi_{\alpha,\beta}^{(S)}$ *and* $\gamma = 1 - (\sum \alpha_i + \sum \beta_j) > 0$, *then* $H_\chi\left(\vartheta_S^\chi\right) = \infty$.

We will prove several subsidiary statements.

Consider the complex type II_1 factor-representation Π_χ of the group $S(\infty)$ which corresponds to the normalize character χ (see (1.1)). We assume that Π_χ is realized in Hilbert space H_χ which is the closure of the linear span of vectors $u \in S(\infty)$ with the scalar product $\langle u, v\rangle_\chi = \chi(uv^*)$. In H_χ we define the unitary representations l_χ and r_χ of the group $S(\infty)$:

$$l_\chi(u)v = uv, \quad r_\chi(u)v = vu^*. \tag{2.1}$$

Let us denote by $\mathcal{L}_\chi$ $(\mathcal{R}_\chi)$ the $W^\star$-algebra generated by $l_\chi\left(S(\infty)\right)$ $\left(r_\chi\left(S(\infty)\right)\right)$ and denote by $H_\chi\left(N_1, N_2, \ldots, N_k\right)$ a CNT-entropy of a system of finite-dimensional subalgebras $N_1, N_2, \ldots, N_k \subset \mathcal{L}_\chi$ (see [5]).

If $\mathfrak{A}$ is an operator family and $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$ then $\mathcal{L}_\chi' = \mathcal{R}_\chi$.

Definition 2.1. *A normalize character χ on G is called an indecomposable one if algebra $\mathcal{L}_\chi$ $(\mathcal{R}_\chi)$ is a factor.*

Lemma 2.1. *Let $\mathcal{A} \subset \mathcal{B}_n$ and let $W^\star$-algebra $\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)$ be generated by operators $l_\chi(g)$ $(g \in S\left(\mathcal{A}\right))$. If χ is an indecomposable normalize character on $S(\infty)$ then*

$$H_\chi(\vartheta_S^\chi) \geq \frac{H_\chi\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right)}{2n+1} \tag{2.2}$$

Proof. Let tr_χ be a trace on $\mathcal{L}_\chi$ that corresponds to character χ. If $\alpha = (\vartheta_S^\chi)^{2n+1}$, $\mathbf{N}_k = \alpha^k\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right)$, then the following properties hold true:

i) $\mathbf{N}_k$ are pairwise commute for any $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers;

ii) if $n_1, n_2 \in \mathbb{Z}$ and $n_1 < n_2$, then $\exists$ a masa [1] $\mathbf{A} \subset \bigvee\limits_{n_1}^{n_2} \mathbf{N}_k$ for which $\mathbf{A}_k = \mathbf{A} \bigcap \mathbf{N}_k$ is the masa in $\mathbf{N}_k$;

iii) $\mathbf{A} = \bigvee\limits_{n_1}^{n_2} \mathbf{A}_k$ and $tr_\chi\left(\prod\limits_{k=n_1}^{n_2} a_k\right) = \prod\limits_{k=n_1}^{n_2} tr_\chi\left(a_k\right) \;\forall\; a_k \in \mathbf{A}_k$.

[1] maximal abelian subalgebra

From these statements and properties (D), (E) [5] it follows that

$$H_\chi\left(\mathbf{N}_{n_1}, \mathbf{N}_{n_1+1}, \ldots, \mathbf{N}_{n_2}\right) = H_\chi\left(\bigvee_{n_1}^{n_2} \mathbf{A}_k\right) = (n_2 - n_1 + 1)\, H_\chi(\mathcal{L}_\chi(S\left(\mathcal{A}\right))).$$

Thus $(2n+1)H_\chi\left(\vartheta_S^\chi\right) = H_\chi(\alpha) \geq H_\chi\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right)$. □

Next statement allows a lower boundary for the entropy $H_\chi(\mathcal{L}_\chi(S(\mathcal{A})))$ in a case of the regular representation.

Lemma 2.2. *Let $\mathcal{A}$ be the same one as in Lemma* 2.1. *If χ is a character of the regular representation, then there exists a number $\mathfrak{C}$ which does not depend on $\mathcal{A}$ and $H_\chi\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right) \geq \mathfrak{C} \cdot |\mathcal{A}| \cdot \ln|\mathcal{A}|$.*

Proof. Let $|\mathcal{A}| = m$, and let $\chi^{(\lambda)}$ be a character of an irreducible representation π_λ of the group $S(m) = S\left(\mathcal{A}\right)$ which corresponds to the Young diagram λ, $\dim\lambda = \dim\pi_\lambda$, $\chi^{(\lambda)}_{norm} = \frac{\chi^{(\lambda)}}{\dim\lambda}$. If χ_m is a restriction of χ on $S(m)$ and $|\lambda|$ is the number of boxes in λ, then

$$\chi_m = \sum_{\lambda:|\lambda|=m} \frac{(\dim\lambda)^2}{m!} \chi^{(\lambda)}_{norm}. \tag{2.3}$$

We denote by e_λ the minimal projection in W^*-algebra $\left(\pi_\lambda\left(S(m)\right)\right)''$ which is generated by operators $\pi_\lambda\left(g\right)$ $(g \in S(m))$. $h(p,q)$ will denote the corresponding hook length for a box $(p,q) \in \lambda$. Recall the well-known hooks-formula

$$\dim\lambda = m! \cdot \prod_{(p,q)\in\lambda} \frac{1}{h(p,q)}. \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\chi_m\left(e_\lambda\right) = \prod_{(p,q)\in\lambda} \frac{1}{h(p,q)}. \tag{2.5}$$

It implies that

$$\begin{aligned} H_\chi\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right) &= \sum_{\lambda:|\lambda|=m} -\dim\lambda \cdot \chi_m\left(e_\lambda\right) \cdot \ln\left(\chi_m\left(e_\lambda\right)\right) \\ &= \sum_{\lambda:|\lambda|=m} \frac{(\dim\lambda)^2}{m!} \cdot \ln\left(\prod_{(p,q)\in\lambda} h(p,q)\right). \end{aligned} \tag{2.6}$$

Using the following inequality belonged to Vershik and Kerov (see [18]) and (2.3)

$$\exp\left[\frac{c_0}{2}\sqrt{m}\right]\cdot\sqrt{m!}\leq\min_{\lambda:|\lambda|=m}\prod_{(p,q)\in\lambda}h(p,q)\leq\exp\left[\frac{c_1}{2}\sqrt{m}\right]\cdot\sqrt{m!},$$

where c_0 and c_1 are positive integers , from (2.6) we obtain

$$H_\chi\left(\mathcal{L}_\chi\left(S\left(\mathcal{A}\right)\right)\right)\geq\frac{c_0}{2}\sqrt{m}+\frac{1}{2}\cdot\ln\left(m!\right).$$

So the statement of our lemma follows from Stirling's formula. $\square$

Now let us take for χ an arbitrary indecomposable normalize character on $S(\infty)$. If $\mathcal{M}$ is an injective finite factor with normalize trace tr, then there is a representation

$$\pi_\chi : S(\infty)\to U(\mathcal{M})$$

with the property

$$\chi(g)=tr\left(\pi_\chi(g)\right).$$

Here $U(\mathcal{M})$ denotes a group of unitary operators in $\mathcal{M}$.

Consider the following operator limit in the weak operator topology

$$\lim_{n\to\infty}\pi_\chi\left((i,n)\right)=A_i, \tag{2.7}$$

where $(i,n)\in S(\infty)$ is a transposition. It is obviously, that $A_i=A_i^*$. Let μ be a spectral measure of operator A_i:

$$\int x^k\mu(dt)=tr\left(A_i^k\right)\ \forall\ k\in\mathbb{N}.$$

We denote by $\mathbb{N}/g$ a set of orbits of a permutation g on the set $\mathbb{N}$. Denote by $|p|$ the cardinality of an orbit $p\in\mathbb{N}/g$. The following statement belongs to A. Okounkov (see [13]).

Lemma 2.3. *The following properties are true:*

a) $A_iA_j=A_jA_i\ \forall\ i,j\in\mathbb{Z}$ *and* $\operatorname{tr}\left(\prod_l A_{j_l}^{k_l}\right)=\prod_l\operatorname{tr}\left(A_{j_l}^{k_l}\right)\ \forall\, k_l\in\mathbb{Z}_+=\mathbb{N}\bigcup\{0\}$*;*

b) $\pi_\chi\left(g\right)A_i\pi_\chi\left(g^{-1}\right)=A_{g(i)}$*;*

c) $\operatorname{supp}\mu\subset[-1,1]$*, the measure* μ *is discrete and* $\forall\,\varepsilon>0$ *a set* $[-1,-\varepsilon]\bigcup[\varepsilon,1]$ *contains at the most* $\frac{2}{\varepsilon}$ *its atoms;*

d) *Let f_i, g_i $(i \in \mathbb{Z})$ are functions on $[-1,1]$ which are pointwise limits of uniformly bounded sequences of continuous functions. If all of f_i, g_i $(i \in \mathbb{Z})$ but finitely many identically equal to* 1, *then*

$$\operatorname{tr}\left(\prod_{i\in\mathbb{Z}} \bar{g}_i\left(A_i\right)\pi_\chi\left(g\right)\prod_{i\in\mathbb{Z}} f_i\left(A_i\right)\right) = \prod_{p\in\mathbb{N}/g}\int x^{|p|-1}\prod_{i\in p} f_i(t)\overline{g_i(t)}\,d\,\mu;$$

e) $\forall\, x \neq 0\ \nu(x) = \frac{\mu(x)}{|x|} \in \mathbb{Z}_+$;

f) *if* $\chi = \chi_{\alpha,\beta}^{(S)}$ *(see* (1.1)*),* $x \neq 0$ *and* $x \in \operatorname{supp}\mu$*, then* $\exists\ i \in \mathbb{N}$*, for which* $\begin{cases}\alpha_i = x, \ldots, \alpha_{i+\nu(x)-1} = x & \text{if } x > 0,\\ \beta_i = |x|, \ldots, \beta_{i+\nu(x)-1} = |x| & \text{if } x < 0.\end{cases}$

Denote by δ_x the function that equals to 1 at the point x, and that equals to 0 at all the rest points. Let $E_i = \delta_0\left(A_i\right)$.

The next statement easily follows from the previous lemma.

Corollary 2.1. *Let* $\chi = \chi_{\alpha,\beta}^{(S)}$, $\gamma = tr\left(E_n\right) = 1 - \sum\limits_i\left(\alpha_i + \beta_i\right)$, *and let* $\mathcal{A}_k = \{i_1, i_2, \ldots, i_k\}$ *be a set of different numbers from* $\mathbb{Z}$*. If* $E_{\mathcal{A}_k} = \prod\limits_{j=1}^{k} E_{i_j}$, $\gamma > 0$, *then for* $g \in S\left(\mathcal{A}\right)$

$$\varphi_{\gamma,k}\left(g\right) = \gamma^{-k}\cdot tr\left(E_{\mathcal{A}_k}\pi_\chi(g)E_{\mathcal{A}_k}\right) = \begin{cases}1 & \text{if } g = e,\\ 0 & \text{otherwise.}\end{cases}$$

From here and from lemma 2.2 it follows the next

Lemma 2.4. *If* $\gamma \in]0,1[$, *and if* E_i $(|i| \leq n)$ *and* $\pi_\chi\left(S\left(\mathcal{B}_n\right)\right)$ *generates a* W^**-algebra* $\mathcal{M}_n$*, then there is some constant* $\mathfrak{C}_1$ *which doesn't depends on* n *and such that*

$$H_\chi\left(\mathcal{M}_n\right) \geq \mathfrak{C}_1\cdot n\ln n.$$

Proof. Let us use the notations of Corollary 2.1. By Lemma 2.2 there exists a constant $\mathfrak{C}$ which does not depend on k and $\mathfrak{C}$ is such that

$$H_{\varphi_{\gamma,k}}\left(E_{\mathcal{A}_k}\pi_\chi\left(S\left(\mathcal{A}_k\right)\right)E_{\mathcal{A}_k}\right) \geq \mathfrak{C}\cdot k\ln k.$$

Taking into consideration this result, we obtain

$$
\begin{aligned}
H_\chi\left(\mathcal{M}_n\right) \geq &\sum_{k=0}^{n} \sum_{\lambda_k:|\lambda_k|=k} \binom{n}{n} (1-\gamma)^{n-k}\gamma^k \frac{(\dim\lambda_k)^2}{k!} \\
&\times \left[\ln\left(\prod_{(p,q)\in\lambda_k} h(p,q)\right) - k\ln\gamma - (n-k)\ln(1-\gamma)\right] \\
&\geq -n\left(\gamma\ln\gamma + (1-\gamma)\ln(1-\gamma)\right) + \mathfrak{C}\sum_{k=0}^{n}\binom{n}{k}(1-\gamma)^{n-k}\gamma^k k\ln k.
\end{aligned}
\tag{2.8}
$$

Now we take a constant $d > 0$ for which

$$
\sum_{k=\left[n\gamma - d\sqrt{n}\right]}^{\left[n\gamma + d\sqrt{n}\right]} \binom{n}{k}(1-\gamma)^{n-k}\gamma^k > \frac{1}{2}\ \forall\, n\in\mathbb{N}.
$$

Taking into account this and (2.8), we have

$$
H_\chi\left(\mathcal{M}_n\right) \geq -n\left(\gamma\ln\gamma + (1-\gamma)\ln(1-\gamma)\right) + \frac{\mathfrak{C}}{2}\left[n\gamma - d\sqrt{n}\right]\ln\left[n\gamma - d\sqrt{n}\right].
$$

Thus, the statement of Lemma 2.4 is proved. □

Proof of Theorem 2.1. If $\gamma = 1$, then the statement of Theorem 2.1 follows from Lemmas 2.1 and 2.2. Let $\gamma < 1$. Using a method we have proved Lemma 2.1, we receive the following estimation

$$
H_\chi\left(\vartheta_S^\chi\right) \geq \frac{H_\chi\left(\mathcal{M}_n\right)}{2n+1} \quad \text{(see Lemma 2.4)}.
$$

Thus, the statement of Theorem 2.1 follows from Lemma 2.4. □

3. The Case of $\gamma = 0$

In this section we will present two different entropy estimation methods developed for the case of finite cardinality of set $I = \{i : \alpha_i > 0\} \cup \{i : \beta_i > 0\}$ and for the case of infinite one correspondingly. First method is based on the important formulaes from the theory of symmetric functions. The second one uses the structural properties of von Neumann factors constructed by the representations of $S(\infty)$. It will be clear, that the case of finite cardinality can be included in the second one, but we would like to show special technic in the Subsection 3.1.

3.1. The Subcase $|I| < \infty$

In this Section we will prove the following theorem.

Theorem 3.1. *Let* $\eta(t) = -t\ln t$, $\sum \alpha_i + \sum \beta_j = 1$, $\chi = \chi^{(S)}_{\alpha,\beta}$ *and let* $\mathcal{N} \in \mathbb{N}$ *exist for which* $\alpha_j = \beta_j = 0\ \forall\ j > \mathcal{N}$. *Then*

$$H_\chi\left(\vartheta_S^\chi\right) = \sum_j \eta\left(\alpha_j\right) + \sum_j \eta\left(\beta_j\right).$$

Consider the restriction of χ onto a finite symmetric group $S(\mathcal{A})$. The characters of the finite symmetric group $S(\mathcal{A})$ are labeled by the Young diagrams with $|\mathcal{A}|$ boxes. Let $\chi^{(\lambda)}$ be a (non normalized) character corresponding to an irreducible representation λ. The restriction $\chi_{S(\mathcal{A})}$ to the group $S(\mathcal{A})$ is a non-negative linear combination of the functions $\chi^{(\lambda)}$

$$\chi\big|_{S(\mathcal{A})} = \sum_{\lambda:|\lambda|=|\mathcal{A}|} \tilde{s}_\lambda(\alpha,\beta)\cdot\chi^{(\lambda)}. \tag{3.1}$$

The Fourier coefficient $\tilde{s}_\lambda(\alpha,\beta)$ is given by the extended Schur function (see [8]), which can be formally defined by Jacoby-Trudi determinant

$$\tilde{s}_\lambda(\alpha,\beta) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+m-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+m-2} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & h_{\lambda_3} & \dots & h_{\lambda_3+m-3} \\ \dots & \dots & \dots & \dots & \dots \\ h_{\lambda_m-m+1} & h_{\lambda_m-m+2} & h_{\lambda_m-m+3} & \dots & h_{\lambda_m} \end{vmatrix}, \tag{3.2}$$

where the *extended complete homogeneous symmetric functions* $h_l = h_l(\alpha,\beta)$ arise as the coefficients of the generating series

$$e^{z\gamma}\prod_{j=1}^{\infty}\frac{1+z\beta_j}{1-z\alpha_j} = 1 + \sum_{l=1}^{\infty} h_l(\alpha,\beta)z^l.$$

We denote by $d = d(\lambda)$ the number of diagonal boxes in the Young diagram λ and we will use the Frobenius notation [10]

$$\lambda = (p_1,\dots,p_d|q_1,\dots,q_d).$$

Here $p_i = \lambda_i - i$ is a number of boxes in the $i-$th row of λ on the right of the $i-$th diagonal box; likewise, $q_i = \lambda'_i - i$ is the number of boxes in the $i-$th column of λ below the $i-$th diagonal box (λ' stands for the transposed diagram).

Lemma 3.1. *Let* $\alpha = \{\alpha_i\}_{i=1}^{\infty}$, $\beta = \{\beta_i\}_{i=1}^{\infty}$ *be Thoma-parameters,* $\sum_{i=1}^{\infty}(\alpha_i+\beta_i) = 1$, $\mathcal{N}_\alpha = \max\{i \in \mathbb{N} : \alpha_k > 0\}$, $\mathcal{N}_\beta = \max\{i \in \mathbb{N} : \beta_k > 0\}$. *If* $\max\{\mathcal{N}_\alpha, \mathcal{N}_\beta\} < \infty$, *then* $s_\lambda(\alpha,\beta) = 0$ *in each of the following cases*

i) $d(\lambda) > d = \max\{\mathcal{N}_\alpha, \mathcal{N}_\beta\}$;

ii) $\lambda_i > d\ \forall\, i = \mathcal{N}_\beta + 1, \ldots, d$;

iii) $\lambda'_i > d\ \forall\ i = \mathcal{N}_\alpha + 1, \ldots, d$.

Proof. We consider a sequence of the Young diagrams

$$\lambda^{(2n+1)} = \left(p_1^{(2n+1)}, \ldots, p_d^{(2n+1)} | q_1^{(2n+1)}, \ldots, q_d^{(2n+1)}\right)$$

with properties:

i) $\left|\lambda^{(2n+1)}\right| = 2n+1$ and $d = d\left(\lambda^{(2n+1)}\right) = \max\{\mathcal{N}_\alpha, \mathcal{N}_\beta\}$ for n sufficiently great;

ii) $\alpha_i = \lim_{n\to\infty} \frac{p_i^{(2n+1)}}{2n+1}$, $\beta_i = \lim_{n\to\infty} \frac{q_i^{(2n+1)}}{2n+1}$ $\quad \forall\ i = 1, \ldots, d$.

It follows from the approximation Theorem [19] that

$$\chi(g) = \chi_{\alpha,\beta}^{(S)}(g) = \lim_{n\to\infty} \frac{\chi^{\left(\lambda^{(2n+1)}\right)}(g)}{\dim \lambda^{(2n+1)}} \quad \forall\ g \in S(\infty).$$

Using this claim, property *i)* and the Young branching rule

$$\chi^{(\Lambda)}\big|_{S(|\Lambda|)} = \sum_{\lambda:\Lambda\searrow\lambda} \chi^{(\lambda)},$$

where the notation $\Lambda \searrow \lambda$ means that diagram $\lambda \subset \Lambda$ is obtained from the diagram Λ by removing a box, we obtain the statement of the lemma. □

Further we will need the Berele-Regev formula (see [1]) for the supersymmetric Schur functions s_λ

$$s_\lambda(x_1, \ldots, x_d; y_1, \ldots, y_d) = \frac{\det\left[x_i^{p_j}\right]_{i,j=1}^d}{V(x_1, \ldots, x_d)} \cdot \frac{\det\left[y_i^{q_j}\right]_{i,j=1}^d}{V(y_1, \ldots, y_d)} \prod_{i,j=1}^{d} (x_i + y_j). \tag{3.3}$$

Here $\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)$, $V(\ldots)$ is the Vandermonde determinant and the parameters $x_1, \ldots, x_d$, as well as $y_1, \ldots, y_d$, are assumed to be pairwise distinct.

If $\sum_{i=1}^{\infty}(\alpha_i+\beta_i)=1$, then the extended Schur (3.1) function coincides with the supersymmetric Schur function

$$\tilde{s}_\lambda(\alpha,\beta)=s_\lambda(\alpha,\beta).$$

Now we will obtain the lower boundary for entropy $H_\chi\left(\mathcal{L}_\chi\left(S\left(n\right)\right)\right)$ (see Lemma 2.1).

Lemma 3.2. *Let parameters* $\alpha=\{\alpha_i\}_{i=1}^{\infty}$ *and* $\beta=\{\beta_i\}_{i=1}^{\infty}$ *satisfy the conditions of Lemma* 3.1, $\chi=\chi_{\alpha,\beta}^{(S)}$. *Then* $\forall\,\varepsilon>0\ \exists\,\mathcal{N}(\varepsilon)\in\mathbb{N}$ *for which*

$$\begin{aligned} H_\chi\left(\mathcal{L}_\chi\left(S\left(n\right)\right)\right)\geq -n(1-\varepsilon)\Bigg\{&\sum_{j=1}^{\mathcal{N}_\alpha}\left[\alpha_j-\varepsilon\right]\cdot\ln\alpha_j\\ &+\sum_{j=1}^{\mathcal{N}_\beta}\left[\beta_j-\varepsilon\right]\cdot\ln\beta_j\Bigg\}+\mathfrak{N}\ln n\quad\forall\,n>\mathcal{N}(\varepsilon), \end{aligned}$$

where $\mathfrak{N}$ *is a constant, which does not depend on* n.

Proof. Let $Y_n(d)$ be a set of Young diagrams λ such, that $|\lambda|=n$ and $d(\lambda)\leq d$. For $k<d$ we define two sets

$$\begin{aligned} Y_n(d,k)&=\left\{\lambda\in Y_n(d):\ \lambda_i'\leq d\ \forall\,i=k+1,k+2,\ldots\right\},\\ Y_n'(d,k)&=\left\{\lambda\in Y_n(d):\ \lambda_i\leq d\ \forall\,i=k+1,k+2,\ldots\right\}. \end{aligned}$$

We assume, that $\mathcal{N}_\alpha\geq\mathcal{N}_\beta$. By Lemma 3.1, we have

$$\chi\big|_{S(n)}=\sum_{\lambda\in Y_n\left(\mathcal{N}_\alpha,\mathcal{N}_\beta\right)}s_\lambda(\alpha,\beta)\cdot\chi^{(\lambda)}. \tag{3.4}$$

Let

$$\begin{aligned} Y_n(d,k,\varepsilon)=\big\{\lambda\in Y_n(d,k):\lambda_i'=\mathcal{N}_\alpha\quad\forall\,i=\mathcal{N}_\beta+1,\ldots,\mathcal{N}_\alpha,\\ |p_i(\lambda)-n\alpha_i|<n\varepsilon\ \text{and}\ |q_j(\lambda)-n\beta_j|<n\varepsilon\\ \forall\,i=1,\ldots,d;\ j=1,\ldots,k\big\}. \end{aligned}\tag{3.5}$$

Using (3.4) and (3.5), by the approximation Theorem [19] we obtain, that there exists $\mathcal{N}(\varepsilon)\in\mathbb{N}$ for which

$$1\geq\sum_{\lambda\in Y_n\left(\mathcal{N}_\alpha,\mathcal{N}_\beta,\varepsilon\right)}\dim\lambda\cdot s_\lambda(\alpha,\beta)>1-\varepsilon\quad\forall\,n>\mathcal{N}(\varepsilon). \tag{3.6}$$

Formula (3.3) can be extended by a continuity to the case, when the number of parameters $x_1, \dots, x_n$ is not equal to the number of parameters $y_1, \dots, y_m$. We assume that, the parameters $\{\alpha_1 \geq \dots \geq \alpha_{\mathcal{N}_\alpha} > 0\}$ are pairwise distinct as well as the parameters $\left\{\beta_1 \geq \dots \geq \beta_{\mathcal{N}_\beta} > 0\right\}$. The next statement is obtained for the diagram $\lambda = (p_1, \dots, p_{\mathcal{N}_\alpha} | q_1, \dots, q_{\mathcal{N}_\alpha}) \in Y_n(\mathcal{N}_\alpha, \mathcal{N}_\beta, \varepsilon)$ from relation (3.3) by passing to the limit ($\beta_{\mathcal{N}_\beta+1} \to 0$, …, $\beta_{\mathcal{N}_\alpha} \to 0$)

$$s_\lambda\left(\alpha_1, \dots, \alpha_{\mathcal{N}_\alpha}; \beta_1, \dots, \beta_{\mathcal{N}_\beta}\right) = \frac{\det\left[\alpha_i^{p_j}\right]_{i,j=1}^{\mathcal{N}_\alpha}}{V\left(\alpha_1, \dots, \alpha_{\mathcal{N}_\alpha}\right)} \times \frac{\det\left[\beta_i^{q_j}\right]_{i,j=1}^{\mathcal{N}_\beta}}{V\left(\beta_1, \dots, \beta_{\mathcal{N}_\beta}\right)} \prod_{i=1}^{\mathcal{N}_\alpha}\left[\alpha_i^{\mathcal{N}_\alpha - \mathcal{N}_\beta} \prod_{j=1}^{\mathcal{N}_\beta}(\alpha_i + \beta_j)\right]. \quad (3.7)$$

Now we consider the case, when there are the coincident parameters. Let $\{n_i(\alpha)\}_{i=1}^{k_\alpha}$ and $\{n_i(\beta)\}_{i=1}^{k_\beta}$ be subsets in $\mathbb{N}$ with the properties:

$$\begin{gathered}\sum_{i=1}^{k_\alpha} n_i(\alpha) = \mathcal{N}_\alpha, \quad \sum_{i=1}^{k_\beta} n_i(\beta) = \mathcal{N}_\beta, \\ \alpha_{n_1(\alpha)+\dots+n_j(\alpha)+1} = \dots = \alpha_{n_1(\alpha)+\dots+n_j(\alpha)+n_{j+1}(\alpha)} = t_j, \\ \beta_{n_1(\beta)+\dots+n_j(\beta)+1} = \dots = \beta_{n_1(\beta)+\dots+n_j(\beta)+n_{j+1}(\beta)} = s_j, \\ \text{the parameters } t_1, \dots, t_{k_\alpha}, \text{are pairwise distinct as well as} \\ s_1, \dots, s_{k_\beta}.\end{gathered} \quad (3.8)$$

If

$$T_{jk} = \begin{cases} t_r^{p_k} & \text{if } j = \sum_{i=1}^{r+1} n_i(\alpha), \\ \prod_{i=1}^{m-1}(p_k - i + 1)\, t_r^{p_k - m + 1} & \text{if } j = -m + \sum_{i=1}^{r+1} n_i(\alpha), \end{cases}$$
$$\text{where } m = 1, \dots, n_{r+1}(\alpha) - 1;$$

$$S_{jk} = \begin{cases} s_r^{q_k} & \text{if } j = 1 + \sum_{i=1}^{r+1} n_i(\beta), \\ \prod_{i=1}^{m-1}(q_k - i + 1)\, s_r^{q_k - m + 1} & \text{if } j = -m + \sum_{i=1}^{r+1} n_i(\beta), \end{cases}$$
$$\text{where } m = 1, \dots, n_{r+1}(\beta) - 1;$$

then we can rewrite (3.7) as follows:

$$s_\lambda(\alpha, \beta) = \frac{\det T}{\prod\limits_{1 \leq l < j \leq k_\alpha} (t_l - t_j)^{n_l(\alpha) \cdot n_j(\alpha)} \cdot (n_j(\alpha) - 1)!\, (n_l(\alpha) - 1)!}$$

$$\times \frac{\det S}{\prod\limits_{1 \le l < j \le k_\beta} (s_l - s_j)^{n_l(\beta) \cdot n_j(\beta)} \cdot (n_j(\beta) - 1)!\,(n_l(\beta) - 1)!}$$

$$\times \prod_{j=1}^{k_\alpha} \prod_{i=1}^{k_\beta} (t_j + s_i)^{n_j(\alpha) \cdot n_i(\beta)}. \quad (3.9)$$

Using inequality

$$x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \ge x_{\pi(1)}^{n_1} x_{\pi(2)}^{n_2} \dots x_{\pi(k)}^{n_k},$$

where π is a permutation, $0 < x_k \le \ldots \le x_2 \le x_1$, $n_i \in \mathbb{N}$ $(1 \le i \le k)$ and $0 < n_k \le \ldots \le n_2 \le n_1$, from (3.9) we have

$$s_\lambda(\alpha, \beta) \ge \frac{P_T(p_1, \ldots, p_{\mathcal{N}_\alpha}) \prod\limits_{i=1}^{k_\alpha} t_i^{n_i(\alpha)\left(\frac{2p_i - n_i(\alpha) + 1}{2}\right)}}{\prod\limits_{1 \le l < j \le k_\alpha} (t_l - t_j)^{n_l(\alpha) \cdot n_j(\alpha)}} \prod_{i=1}^{\mathcal{N}_\alpha} \left[\alpha_i^{\mathcal{N}_\alpha - \mathcal{N}_\beta}\right]$$

$$\times \frac{P_S(q_1, \ldots, q_{\mathcal{N}_\beta}) \prod\limits_{i=1}^{k_\beta} s_i^{n_i(\beta)\left(\frac{2p_i - n_i(\beta) + 1}{2}\right)}}{\prod\limits_{1 \le l < j \le k_\beta} (s_l - s_j)^{n_l(\beta) \cdot n_j(\beta)}} \cdot \prod_{j=1}^{k_\alpha} \prod_{i=1}^{k_\beta} (t_j + s_i)^{n_j(\alpha) \cdot n_i(\beta)}. \quad (3.10)$$

Here $P_T\,(P_S)$ is a polynomial of $\mathcal{N}_\alpha - k_\alpha$ $(\mathcal{N}_\beta - k_\beta)$ degree with coefficients which does not depend on n. Thus, we have

$$\begin{aligned} H_\chi(\mathcal{L}_\chi(S(n))) &= -\sum_{\lambda : |\lambda| = n} \dim\lambda \cdot s_\lambda(\alpha, \beta) \cdot \ln s_\lambda(\alpha, \beta) \\ &\ge -\sum_{\lambda \in Y_n(\mathcal{N}_\alpha, \mathcal{N}_\beta)} \dim\lambda \cdot s_\lambda(\alpha, \beta) \cdot \ln s_\lambda(\alpha, \beta) \\ &\overset{\text{see (3.10), (3.8)}}{\ge} -\sum_{\lambda \in Y_n(\mathcal{N}_\alpha, \mathcal{N}_\beta)} \dim\lambda \cdot s_\lambda(\alpha, \beta) \end{aligned}$$

$$\times \left[\left(\sum_{j=1}^{\mathcal{N}_\alpha} p_i \cdot \ln\alpha_i + \sum_{j=1}^{\mathcal{N}_\beta} q_i \cdot \ln\beta_i\right) + (\mathcal{N}_\alpha + \mathcal{N}_\beta - k_\alpha - k_\beta)\ln n + \mathcal{C}(\alpha, \beta)\right].$$

Here $\mathcal{C}(\alpha, \beta)$ is a constant that does not depend on n. From here, taking into account (3.5) and (3.6), we obtain the statement of the lemma. The case when $\mathcal{N}_\alpha < \mathcal{N}_\beta$ can be considered analogous by taking $Y_n(\cdot, \cdot)$ instead of $Y_n'(\cdot, \cdot)$. $\square$

3.2. The Subcase of Infinite Cardinality

Next we consider the case of the infinite number of nonzero parameters $\{\alpha_i\} = \alpha$, $\{\beta_i\} = \beta$ and obtain a lower boundary for the entropy $H_\chi(\mathcal{M}_n)$, where $\mathcal{M}_n$ is generated by A_0 and $\pi_\chi(S(n))$ as a W^*-algebra.

Lemma 3.3. *If* $\sum \alpha_i + \sum \beta_j = 1$, *then*

$$H_\chi\left(\vartheta_S^\chi\right) \geq \sum_i \Big(\frac{\eta\left(\nu(\alpha_i)\cdot\alpha_i\right)}{\nu(\alpha_i)} + \frac{\eta\left(\nu(\beta_i)\cdot\beta_i\right)}{\nu(\beta_i)}\Big),$$

where ν is the multiplicity function (see Lemma 2.3).

Proof. Let $\chi = \chi_{\alpha,\beta}^{(S)}$ and let π_χ be the representation that corresponds to χ. We denote by $\mathfrak{A}$ the W^*-algebra which is generated by operators $\{A_i\}_{i\in\mathbb{Z}}$ (see Lemma 2.3). Since $\vartheta_S^\chi(A_i) = A_{i+1}$ (2.7), ϑ_S^χ restricts to an automorphism of $\mathfrak{A}$. So we get, using properties *a*), *c*), *d*), *e*), *f*) of Lemma 2.3, that the Abelian dynamical system $\left(\mathfrak{A}, \vartheta_S^\chi, tr\right)$ is the classical Bernoulli shift with the entropy

$$\sum_i \Big(\frac{\eta\left(\nu(\alpha_i)\cdot\alpha_i\right)}{\nu(\alpha_i)} + \frac{\eta\left(\nu(\beta_i)\cdot\beta_i\right)}{\nu(\beta_i)}\Big).$$

□

Let us consider the following union $\{\alpha_i\} = \bigcup_j U_j$, where $\forall\,\alpha_k, \alpha_l \in U_j,\ \alpha_k = \alpha_l$, $\forall\,\alpha_k \in U_j,\ \alpha_l \in U_m,\ \alpha_k > \alpha_l$ if $j < m$. Next we define $\alpha' = \{\alpha_i'\}$ such that $\forall\, i\ \alpha_i' \in U_i,\ \alpha_i' \neq 0$ and $\forall\, i, j,\ i \neq j\ \alpha_i' \neq \alpha_j'$. In the same way we define the sequence $\beta' = \{\beta_i'\}$.

Let $a_i = \nu\left(\alpha_i'\right)\alpha_i'$, $b_i = \nu\left(\beta_i'\right)\beta_i'$ and let

$$\begin{aligned}
&\mathcal{N}_{\alpha,\beta}\left(m, k_\alpha, k_\beta, D\right) \\
&= \Big\{\left(m_1^\alpha, \ldots, m_{k_\alpha}^\alpha, m_1^\beta, \ldots, m_{k_\beta}^\beta, m_{k_\alpha+k_\beta+1}\right) \in \mathop{\times}_{j=1}^{k_\alpha+k_\beta+1} \mathbb{N} : \\
&\left(a_i m - D\sqrt{m} \leq m_i^\alpha \leq a_i m + D\sqrt{m}\right) \\
&\bigwedge \left(b_j m - D\sqrt{m} \leq m_j^\beta \leq b_j m + D\sqrt{m}\right) \\
&\bigwedge \Big(\sum_{j=1}^{k_\alpha} m_j^\alpha + \sum_{j=1}^{k_\beta} m_j^\beta + m_{k_\alpha+k_\beta+1} = m\Big) \\
&\forall\, i = 1, 2\ldots, k_\alpha;\ j = 1, 2\ldots, k_\beta\Big\}.
\end{aligned} \tag{3.11}$$

The next statements follows from the central limit theorem.

Lemma 3.4. *Let* $\sum \alpha_i + \sum \beta_j = 1$, $a_j = \nu\left(\alpha'_j\right)\alpha'_j$, $b_j = \nu\left(\beta'_j\right)\beta'_j$, $\gamma_{kl} = 1 - \sum\limits_{j=1}^{k} a_j - \sum\limits_{j=1}^{l} b_j$ *and let* δ_1, δ_2 *be given. Then there are* $\mathrm{N}(\delta_1)$, $\mathrm{N}(\delta_1,\delta_2) \in \mathbb{N}$ *and* $D = D(\delta_1,\delta_2) > 0$ *with properties:*

i) $\gamma_{kl} < \delta_1 \ \forall\, k,l \geq \mathrm{N}(\delta_1)$;

ii) if $k_\alpha = \min\{\mathrm{N}(\delta_1), |\alpha'|\}$, $k_\beta = \min\{\mathrm{N}(\delta_1), |\beta'|\}$, $m \geq \mathrm{N}(\delta_1,\delta_2)$, *then*

$$\sum_{\widetilde{m}\in\mathcal{N}_{\alpha,\beta}\left(m,k_\alpha,k_\beta,D\right)} \frac{m! \cdot \prod\limits_{j=1}^{k_\alpha} a_j^{m_j^\alpha} \prod\limits_{j=1}^{k_\beta} b_j^{m_j^\beta}}{m_{k_\alpha+k_\beta+1}! \cdot \prod\limits_{j=1}^{k_\alpha} m_j^\alpha! \prod\limits_{j=1}^{k_\beta} m_j^\beta!} \cdot \gamma_{k_\alpha k_\beta}^{\;m_{k_\alpha+k_\beta+1}} > 1-\delta_2,$$

where $\widetilde{m} = \left(m_1^\alpha,\dots,m_{k_\alpha}^\alpha, m_1^\beta,\dots,m_{k_\beta}^\beta, m_{k_\alpha+k_\beta+1}\right)$ *and* D, $\mathrm{N}(\delta_1,\delta_2)$ *are constants which do not depend on* m.

Let $E_i(x) = \delta_x(A_i)$ (see Lemma 2.3) and let $\mathcal{A}_k = \{i_1, i_2, \dots, i_k\}$ be a set of different numbers from $\mathbb{Z}$. We denote by $E_{\mathcal{A}_k}(x)$ projection $\prod\limits_{j=1}^{k} E_{i_j}(x)$. If $g \in S(\mathcal{A}_k)$ then by Lemma 2.3 *b)*

$$[E_{\mathcal{A}_k}(x), \pi_\chi(g)] = 0. \tag{3.12}$$

Therefore, the positive definite function $\tau_{\mathcal{A}_k}$ on $S(\mathcal{A}_k)$, which is defined by formula

$$\tau_{\mathcal{A}_k,x}(g) = \frac{tr\left(E_{\mathcal{A}_k}(x)\pi_\chi(g)\right)}{tr\left(E_{\mathcal{A}_k}(x)\right)}, \tag{3.13}$$

where $x \in \{\alpha'\} \bigcup \{\beta'\}$, is the normalize character.

The next Lemma is an auxiliary one.

Lemma 3.5. *The next "dual" formula for extended Schur functions*

$$\tilde{s}_\lambda(\alpha,\beta) = \tilde{s}_{\lambda'}(\beta,\alpha), \tag{3.14}$$

where λ' *is the transposed diagram for* λ, *is valid.*

Proof. The formula (3.14) is the generalization of the formula (2.9′) from [10]. We will repeat the main ideas of that proof as applied to our case. Let us denote

$$H_{(\alpha,\beta)}(z) = e^{z\gamma} \prod_{j=1}^{\infty} \frac{1+z\beta_j}{1-z\alpha_j} = 1 + \sum_{l=1}^{\infty} h_l(\alpha,\beta) z^l. \tag{3.15}$$

Then

$$H_{(\alpha,\beta)}(z)H_{(\beta,\alpha)}(-z) \equiv 1. \tag{3.16}$$

Let us consider two matrices

$$\mathcal{H} = (h_{i-j}(\alpha,\beta))_{0\le i,j\le N} \tag{3.17}$$

and

$$\tilde{\mathcal{H}} = \left((-1)^{i-j}h_{i-j}(\beta,\alpha)\right)_{0\le i,j\le N}, \tag{3.18}$$

where N is some positive integer. We remind that $h_k(\alpha,\beta) = 0$ for $k < 0$ and hence the both matrices are upper-triangular with

$$\det\tilde{\mathcal{H}} = \det\mathcal{H} = 1. \tag{3.19}$$

Moreover, in view of (3.16)

$$\tilde{\mathcal{H}}\mathcal{H} = \mathcal{H}\tilde{\mathcal{H}} = I \tag{3.20}$$

holds. Hence $\tilde{\mathcal{H}} = \mathcal{H}^{-1}$. Let $\tilde{\mathcal{H}}'$ be the transposed matrix for $\tilde{H}$, M is an arbitrary minor of the matrix $\mathcal{H}$ and A is the algebraic adjunct corresponding to the minor M' of the matrix $\tilde{\mathcal{H}}'$ with the same numbers of columns and rows as the numbers of ones in M. By the Laplace theorem and by the (3.19)-(3.20) we obtain the equation $M = A$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a Young diagram, $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ be the transposed diagram. Then by the (3.2) $\tilde{s}_\lambda(\alpha,\beta)$ can be consider as the minor of the matrix $\mathcal{H}$ with the raw numbers $\lambda_i - i + n$, $1 \le i \le n$ and the column numbers $n-j$, $1 \le j \le n$. It is well-known that the $m+n$ numbers $\lambda_i - i + n$, $1 \le i \le n$ and $(m+n-1)-(\lambda'_j - j + m) = n - 1 - \lambda'_j + j$, $1 \le j \le m$ are the permutation of the $\{0, 1, 2, \ldots, m+n-1\}$ (see [10]). Below we assume that the dimension of the matrixes $N = m+n-1$. Then the corresponded algebraic adjunct has the raw numbers $n-1-\lambda'_i+i$, $1 \le i \le m$ and the column numbers $n-1+j$, $1 \le j \le m$. Since the elements of the matrix $\tilde{\mathcal{H}}'$ look like $(-1)^{j-i}h_{j-i}(\beta,\alpha)$ the algebraic adjunct consists of such elements $(-1)^{\lambda'_i+j-i}h_{\lambda'_i+j-i}(\beta,\alpha)$. Besides $\sum_{i=1}^n(\lambda_i - i + n) - \sum_{j=1}^n(n-j) = |\lambda|$. Thus

$$\begin{aligned}\tilde{s}_\lambda(\alpha,\beta) &= \det\left(h_{\lambda+j-i}(\alpha,\beta)\right)_{1\le i,j\le n} \\ &= (-1)^{|\lambda|}\det\left((-1)^{\lambda'_i+j-i}h_{\lambda'+j-i}(\beta,\alpha)\right)_{1\le i,j\le m} \\ &= \det\left(h_{\lambda'+j-i}(\beta,\alpha)\right)_{1\le i,j\le m} = \tilde{s}_{\lambda'}(\beta,\alpha),\end{aligned}$$

$\square$

Lemma 3.6. *Let $\mathfrak{l}(g)$ $(g \in S(\mathcal{A}_k))$ is the number of cycles of a permutation g. Then*

$$\tau_{\mathcal{A}_k,x}(g) = \frac{(\operatorname{sign} x)^{k-\mathfrak{l}(g)}}{\nu(x)^{k-\mathfrak{l}(g)}}. \tag{3.21}$$

Therefore, $\tau_{\mathcal{A}_k,x}$ is the restriction of characters $\chi^{(S)}_{\alpha_{\nu(x)},0}$ for $x \in \alpha'$ $\left(\chi^{(S)}_{0,\beta_{\nu(x)}}\right.$ for $\left.x \in \beta'\right)$ to the group $S(\mathcal{A}_k)$. Here $\alpha_\nu, \beta_\nu = \{\underbrace{\nu^{-1}, \ldots, \nu^{-1}}_{\nu}\}$.

Proof. We denote by $\mathcal{A}_k/g$ a set of orbits of the permutation g on the set $\mathcal{A}_k$. If μ is a spectral measure of operator A_i, then $\mu(x) = \nu(x) \cdot |x|$ (Lemma 2.3 *e*)) and by (Lemma 2.3 *d*)) we obtain

$$\tau_{\mathcal{A}_k,x}(g) = \frac{\prod\limits_{p \in \mathcal{A}_k/g} \left[x^{|p|-1}\mu(x)\right]}{|x^k|\nu^k(x)} = \frac{x^{k-\mathfrak{l}(g)}|x|^{\mathfrak{l}(g)}\nu^{\mathfrak{l}(g)}(x)}{|x|^k \cdot \nu^k(x)} = \frac{(sign\, x)^{k-\mathfrak{l}(g)}}{\nu^{k-\mathfrak{l}(g)}(x)}.$$

□

Let parameters $\alpha = \{\alpha_i\}_{i=1}^\infty$ and $\beta = \{\beta_i\}_{i=1}^\infty$ satisfy the conditions of Lemma 3.2, $\chi = \chi^{(S)}_{\alpha,\beta}$, the W^*-algebra $\mathcal{L}_\chi(S(\mathcal{A}_k))$ be generated by operators $\pi_\chi(S(\mathcal{A}_k))$. We denote by $\mathcal{C}^x_k(\mathcal{A}_k)$ the center of the W^*-algebra $\mathfrak{M}^x_k(\mathcal{A}_k) = E_{\mathcal{A}_k}(x)\mathcal{L}_\chi(S(\mathcal{A}_k))$.

At first we assume that $x > 0$. Then from (3.1) and Lemma 3.4 we obtain

$$\tau_{\mathcal{A}_k,x} = \chi^{(S)}_{\alpha_{(\nu(x))},0}\big|_{S(\mathcal{A}_k)} = \sum_{\lambda:|\lambda|=k} \tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) \cdot \chi^{(\lambda)}. \tag{3.22}$$

The coefficients $\tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right)$ in the expansion can be easily evaluated by using (3.1)

$$\tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) = |\nu(x)|^{-k} \begin{vmatrix} \binom{\nu+\lambda_1-1}{\nu-1} & \binom{\nu+\lambda_1}{\nu-1} & \ldots & \binom{2\nu+\lambda_1-2}{\nu-1} \\ \binom{\nu+\lambda_2-2}{\nu-1} & \binom{\nu+\lambda_2-1}{\nu-1} & \ldots & \binom{2\nu+\lambda_2-3}{\nu-1} \\ \ldots & \ldots & \ldots & \ldots \\ \binom{\lambda_\nu}{\nu-1} & \binom{\lambda_\nu+1}{\nu-1} & \ldots & \binom{\nu+\lambda_\nu-1}{\nu-1} \end{vmatrix}. \tag{3.23}$$

If $x < 0$, then

$\tau_{\mathcal{A}_k,x} = \chi^{(S)}_{0,\beta_{\nu(x)}}\big|_{S(\mathcal{A}_k)} = \sum\limits_{\lambda:|\lambda|=k} \tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) \cdot \chi^{(\lambda)}$ and by the Lemma 3.5

$$\tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) = \tilde{s}_{\lambda'}\left(\beta_{\nu(x)}, 0\right) = |\nu(x)|^{-k} \begin{vmatrix} \binom{\nu+\lambda'_1-1}{\nu-1} & \binom{\nu+\lambda'_1}{\nu-1} & \ldots & \binom{2\nu+\lambda'_1-2}{\nu-1} \\ \binom{\nu+\lambda'_2-2}{\nu-1} & \binom{\nu+\lambda'_2-1}{\nu-1} & \ldots & \binom{2\nu+\lambda'_2-3}{\nu-1} \\ \ldots & \ldots & \ldots & \ldots \\ \binom{\lambda'_\nu}{\nu-1} & \binom{\lambda'_\nu+1}{\nu-1} & \ldots & \binom{\nu+\lambda'_\nu-1}{\nu-1} \end{vmatrix}. \tag{3.24}$$

Here λ' stands for the transposed diagram and $\nu = \nu(x)$.

Let Y_k be a set of a Young diagrams λ such that $|\lambda| = k$. We introduce further the set $Y_k(x) = \begin{cases} \left\{\lambda \in Y_k : \tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) \neq 0\right\} & \text{if } x > 0, \\ \left\{\lambda \in Y_k : \tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) \neq 0\right\} & \text{if } x < 0 \end{cases}$ and denote by $\mathfrak{S}_k^x(\mathcal{A}_k)$ the set of all minimal projections in $\mathcal{C}_k^x(\mathcal{A}_k)$. By virtue of (3.22) and(3.24), the mapping

$$\lambda \in Y_k(x) \to \frac{\dim \lambda}{k!} E_{\mathcal{A}_k}(x) \cdot \sum_{g \in S(\mathcal{A}_k)} \chi^{(\lambda)}(g) \pi_\chi(g) = \mathfrak{e}_k^x(\lambda) \in \mathfrak{S}_k^x(\mathcal{A}_k)$$

is one-to-one correspondence and true

Lemma 3.7. *Let $\mathfrak{S}_{k\,\lambda}^x(\mathcal{A}_k)$ be the set of all minimal projections in $\mathfrak{e}_k^x(\lambda)\mathfrak{M}_k^x(\mathcal{A}_k)$. If $\mathfrak{e} \in \mathfrak{S}_{k\,\lambda}^x(\mathcal{A}_k)$ then*

$$\chi_{\alpha,\beta}^{(S)}(\mathfrak{e}) = [\nu(x)|x|]^k \cdot \begin{cases} \tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) & \text{if } x > 0, \\ \tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) & \text{if } x < 0 \end{cases} = \mathfrak{o}_x^\lambda(k)|x|^k,$$

where $\mathfrak{o}_x^\lambda(k) \in \mathbb{N}\ \forall\ k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} \dfrac{\mathfrak{o}_x^\lambda(k)}{k^{\nu^2(x)}} = 0$ uniformly on the set Y_k.

Proof. Using (3.22), (3.23) and (3.24), we obtain the statement of Lemma 3.7 from the next chain of equalities

$$\chi_{\alpha,\beta}^{(S)}(\mathfrak{e}) = \chi_{\alpha,\beta}^{(S)}(E_{\mathcal{A}_k}(x)) \cdot \tau_{\mathcal{A}_k,x}(\mathfrak{e}) = [\nu(x)|x|]^k \cdot \begin{cases} \tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) & \text{if } x > 0, \\ \tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) & \text{if } x < 0. \end{cases}$$

Let us denote for $x > 0$

$$\mathfrak{o}_x^\lambda(k) = \begin{vmatrix} \binom{\nu+\lambda_1-1}{\nu-1} & \binom{\nu+\lambda_1}{\nu-1} & \dots & \binom{2\nu+\lambda_1-2}{\nu-1} \\ \binom{\nu+\lambda_2-2}{\nu-1} & \binom{\nu+\lambda_2-1}{\nu-1} & \dots & \binom{2\nu+\lambda_2-3}{\nu-1} \\ \dots & \dots & \dots & \dots \\ \binom{\lambda_\nu}{\nu-1} & \binom{\lambda_\nu+1}{\nu-1} & \dots & \binom{\nu+\lambda_\nu-1}{\nu-1} \end{vmatrix}$$

and for $x < 0$

$$\mathfrak{o}_x^\lambda(k) = \begin{vmatrix} \binom{\nu+\lambda'_1-1}{\nu-1} & \binom{\nu+\lambda'_1}{\nu-1} & \dots & \binom{2\nu+\lambda'_1-2}{\nu-1} \\ \binom{\nu+\lambda'_2-2}{\nu-1} & \binom{\nu+\lambda'_2-1}{\nu-1} & \dots & \binom{2\nu+\lambda'_2-3}{\nu-1} \\ \dots & \dots & \dots & \dots \\ \binom{\lambda'_\nu}{\nu-1} & \binom{\lambda'_\nu+1}{\nu-1} & \dots & \binom{\nu+\lambda'_\nu-1}{\nu-1} \end{vmatrix}.$$

The function $\mathfrak{o}_x^\lambda(k)$ can be considered as a polynomial from $\nu(x)$ variables $((\lambda_1, \lambda_2, \dots, \lambda_\nu)$ or $(\lambda'_1, \lambda'_2, \dots, \lambda'_\nu))$ with the degree equals $\nu(x)(\nu(x)-1)$. Thus the lemma is proved. $\square$

By means of $\widetilde{m} \in \mathcal{N}_{\alpha,\beta}(m, k_\alpha, k_\beta, D)$ (see (3.11), p. 27) we introduce pairwise disjoint subsets $\mathcal{Q}\left(\alpha'_j\right)$ $(1 \le j \le k_\alpha)$ and $\mathcal{Q}\left(\beta'_j\right)$ $(1 \le j \le k_\beta)$ in $\mathcal{I}(m) = \{1, 2, \ldots, m\}$ with properties

$$\left|\mathcal{Q}\left(\alpha'_j\right)\right| = m_j^\alpha,\ \left|\mathcal{Q}\left(\beta'_j\right)\right| = m_j^\beta.$$

Let $\mathcal{Q}(\gamma) = \mathcal{I}(m) \setminus \left(\bigcup_{j=1}^{k_\alpha} \mathcal{Q}(\alpha'_j) \bigcup_{j=1}^{k_\beta} \mathcal{Q}(\beta'_j)\right)$ and

$$E_{\widetilde{\mathcal{Q}}(\tilde{m})} = \left[\prod_{j=1}^{k_\alpha}\prod_{l=1}^{k_\beta}\prod_{s\in\mathcal{Q}(\alpha'_j)} E_s\left(\alpha'_j\right) \prod_{s\in\mathcal{Q}(\beta'_l)} E_s\left(\beta'_l\right)\right] \prod_{s\in\mathcal{Q}(\gamma)} F_s, \tag{3.25}$$

where $F_s = \mathbb{I} - \sum_{j=1}^{k_\alpha} E_s(\alpha'_j) - \sum_{j=1}^{k_\beta} E_s(\beta'_j)$, $\widetilde{\mathcal{Q}}(\tilde{m})$ is an ordered set $\left(\mathcal{Q}(\alpha'_1), \ldots, \mathcal{Q}(\alpha'_{k_\alpha}); \mathcal{Q}(\beta'_1), \ldots, \mathcal{Q}(\beta'_{k_\beta}); \mathcal{Q}(\gamma)\right)$. If

$$G\big(\widetilde{\mathcal{Q}}(\tilde{m})\big) = \mathop{\times}_{j=1}^{k_\alpha} S\left(\mathcal{Q}\left(\alpha'_j\right)\right) \mathop{\times}_{i=1}^{k_\beta} S\left(\mathcal{Q}\left(\beta'_i\right)\right)$$

and $\mathcal{L}_\chi\big(G\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)\big)$ is generated by operators $\pi_\chi\big(G\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)\big)$ as a W^*-algebra, then $\mathcal{M}\big(\widetilde{\mathcal{Q}}(\tilde{m})\big) = E_{\widetilde{\mathcal{Q}}(\tilde{m})}\mathcal{L}_\chi\big(G\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)\big)$ is isomorphic to

$$\bigotimes_{j=1}^{k_\alpha} \mathfrak{M}_{m_j^\alpha}^{\alpha'_j}\left(\mathcal{Q}\left(\alpha'_j\right)\right) \bigotimes_{i=1}^{k_\beta} \mathfrak{M}_{m_i^\beta}^{\beta'_i}\left(\mathcal{Q}\left(\beta'_i\right)\right) \text{ (see p.30).} \tag{3.26}$$

Lemma 3.8. *Let tr_χ is the central normalize state on $\mathcal{L}_\chi(S(\infty))$ which corresponds to $\chi = \chi_{\alpha,\beta}^{(S)}$. If $\varphi_{\tilde{m}}$ is the restriction tr_χ to the algebra $\mathcal{M}\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)$ and $H_{\varphi_{\tilde{m}}}\big(\mathcal{M}\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)\big)$ is the CNT-entropy of $\mathcal{M}\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)$ corresponding to $\varphi_{\tilde{m}}$, then*

$$H_{\varphi_{\tilde{m}}}\big(\mathcal{M}\big(\widetilde{\mathcal{Q}}(\tilde{m})\big)\big) = -\gamma_{k_\alpha k_\beta}^{m_{k_\alpha+k_\beta+1}} \prod_{j=1}^{k_\alpha} a_j^{m_j^\alpha} \prod_{j=1}^{k_\beta} b_j^{m_j^\beta}$$
$$\times\left\{\left(m_{k_\alpha+k_\beta+1}\right)\ln\left(\gamma_{k_\alpha k_\beta}\right) + \sum_{j=1}^{k_\alpha} m_j^\alpha \ln\alpha'_j + \sum_{j=1}^{k_\beta} m_j^\beta \ln\beta'_j + \mathcal{O}(\ln m)\right\},$$

where $0 \le \limsup_{m\to\infty} \frac{\mathcal{O}(\ln m)}{\ln m} < \infty$, $\gamma_{k_\alpha k_\beta} = \chi(F_s) = 1 - \sum_{j=1}^{k_\alpha} a_j - \sum_{j=1}^{k_\beta} b_j$.

Proof. We denote by $\varphi^{x}_{\mathcal{A}_k}$ the restriction tr_χ to the algebra $\mathfrak{M}^x_k\left(\mathcal{A}_k\right)$. In view of (3.25) and (3.26), we have

$$\varphi_{\tilde{m}} = \gamma_{k_\alpha\, k_\beta}^{m_{k_\alpha+k_\beta+1}} \bigotimes_{j=1}^{k_\alpha} \varphi_{\mathcal{Q}(\alpha'_j)}^{\alpha'_j} \bigotimes_{i=1}^{k_\beta} \varphi_{\mathcal{Q}(\beta'_i)}^{\beta'_i}. \tag{3.27}$$

Let

$$f_x^\lambda = \begin{cases} \tilde{s}_\lambda\left(\alpha_{\nu(x)}, 0\right) & \text{if } x > 0, \\ \tilde{s}_\lambda\left(0, \beta_{\nu(x)}\right) & \text{if } x < 0. \end{cases}$$

Further, using (3.22) and (3.24), we obtain

$$\begin{aligned} &H_{\varphi^x_{\mathcal{A}_k}}\left(\mathfrak{M}^x_k\left(\mathcal{A}_k\right)\right) \\ &\quad = -\left(\nu(x)|x|\right)^k \left\{ \sum_{\lambda:|\lambda|=k} \dim\lambda \cdot f_x^\lambda \left[\ln f_x^\lambda + \ln\left(\left(\nu(x)|x|\right)^k\right)\right]\right\} \\ &\quad = -\left(\nu(x)|x|\right)^k \left\{ \sum_{\lambda:|\lambda|=k} \dim\lambda \cdot f_x^\lambda \left[k\ln|x| + \ln\left(\left(\nu(x)\right)^k f_x\right)\right]\right\} \\ &\overset{\text{(see Lemma 3.7)}}{=} -\left(\nu(x)|x|\right)^k \left\{ \sum_{\lambda:|\lambda|=k} \dim\lambda \cdot f_x^\lambda \left[k\ln|x| + \ln\left(\mathfrak{o}_x^\lambda(k)\right)\right]\right\}. \end{aligned}$$

Since $\sum\limits_{\lambda:|\lambda|=k} \dim\lambda \cdot f_x^\lambda = 1$, we may rewrite $H_{\varphi^x_{\mathcal{A}_k}}\left(\mathfrak{M}^x_k\left(\mathcal{A}_k\right)\right)$ as follows:

$$H_{\varphi^x_{\mathcal{A}_k}}\left(\mathfrak{M}^x_k\left(\mathcal{A}_k\right)\right) = -k\left(\nu(x)|x|\right)^k \ln|x| - \left(\nu(x)|x|\right)^k \mathfrak{O}\left(x, \ln k\right), \tag{3.28}$$

where $\mathfrak{O}\left(x, \ln k\right) = \sum\limits_{\lambda:|\lambda|=k} \dim\lambda \cdot f_x^\lambda \ln\left(\mathfrak{o}_x^\lambda(k)\right)$ and by Lemma 3.7

$$0 \le \limsup_{k\to\infty} \frac{\mathfrak{O}\left(x, \ln k\right)}{\ln k} < \infty. \tag{3.29}$$

It follows from (3.26), (3.27) and (3.28) that

$$\begin{aligned} H_{\varphi_{\tilde{m}}}\left(\mathcal{M}\left(\widetilde{\mathcal{Q}}\left(\tilde{m}\right)\right)\right) &\overset{\text{(by definition } \varphi_{\tilde{m}})}{=} -\gamma_{k_\alpha\, k_\beta}^{m_{k_\alpha+k_\beta+1}} \prod_{j=1}^{k_\alpha} a_j^{m_j^\alpha} \prod_{j=1}^{k_\beta} b_j^{m_j^\beta} \\ &\times \left\{ \left(m_{k_\alpha+k_\beta+1}\right)\ln\left(\gamma_{k_\alpha\, k_\beta}\right) + \sum_{j=1}^{k_\alpha} m_j^\alpha \ln\alpha'_j + \sum_{j=1}^{k_\beta} m_j^\beta \ln\beta'_j \right. \\ &\left. + \sum_{j=1}^{k_\alpha} \mathfrak{O}\left(\alpha'_j, \ln m_j^\alpha\right) + \sum_{j=1}^{k_\beta} \mathfrak{O}\left(\beta'_j, \ln m_j^\beta\right)\right\}. \end{aligned}$$

Using (3.29) and the equality $\sum\limits_{j=1}^{k_\alpha} m_j^\alpha + \sum\limits_{j=1}^{k_\beta} m_j^\beta + m_{k_\alpha+k_\beta+1} = m$ in latter assertion, we have

$$0 \le \limsup_{m\to\infty} \frac{\sum\limits_{j=1}^{k_\alpha} \mathfrak{O}\big(\alpha_j', \ln m_j^\alpha\big) + \sum\limits_{j=1}^{k_\beta} \mathfrak{O}\big(\beta_j', \ln m_j^\beta\big)}{\ln m} < \infty$$

and

$$H_{\varphi_{\tilde m}}\big(\mathcal{M}\big(\widetilde{\mathcal{Q}}\,(\tilde m)\big)\big) = -\gamma_{k_\alpha\, k_\beta}^{m_{k_\alpha+k_\beta+1}} \prod_{j=1}^{k_\alpha} a_j^{m_j^\alpha} \prod_{j=1}^{k_\beta} b_j^{m_j^\beta}$$
$$\times \left\{ \left(m_{k_\alpha+k_\beta+1}\right) \ln\left(\gamma_{k_\alpha\, k_\beta}\right) + \sum_{j=1}^{k_\alpha} m_j^\alpha \ln \alpha_j' + \sum_{j=1}^{k_\beta} m_j^\beta \ln \beta_j' + \mathcal{O}\,(\ln m) \right\}.$$

□

In the next statements we will use the notations from Lemma 3.4.

Proposition 3.1. *Let $\mathcal{M}_m$ be a W^*-algebra generated by A_0 and $\pi_\chi\,(S(m))$ and let δ_1, δ_2 be given. Then there exists a natural numbers: $k_\alpha\,(\delta_1)$, $k_\beta\,(\delta_1)$, $\mathrm{C}\,(\delta_1,\delta_2)$, and $\mathrm{M}\,(\delta_1,\delta_2)$ such that*

i) $\gamma_{k_\alpha k_\beta} = 1 - \sum\limits_{j=1}^{k_\alpha(\delta_1)} a_j - \sum\limits_{j=1}^{k_\beta(\delta_1)} b_j < \delta_1$;

ii) $\forall\ m > \mathrm{M}\,(\delta_1,\delta_2)$

$$H_\chi\,(\mathcal{M}_m) \ge -m\,(1-\delta_2) \left\{ \sum_{j=1}^{k_\alpha(\delta_1)} \nu\left(\alpha_j'\right) \alpha_j' \ln \alpha_j' \right.$$
$$\left. + \sum_{j=1}^{k_\beta(\delta_1)} \nu\left(\beta_j'\right) \beta_j' \ln \beta_j' \right\} + \mathrm{C}\,(\delta_1,\delta_2)\sqrt{m} \quad (\textit{see Lemma 3.8}).$$

Proof. It is clear that $\mathcal{M}\big(\widetilde{\mathcal{Q}}\,(\tilde m)\big) \subset \mathcal{M}_m$. Therefore,

$$H_\chi\,(\mathcal{M}_m) \ge \sum_{\tilde m} \frac{m! \cdot H_{\varphi_{\tilde m}}\left(\mathcal{M}\left(\widetilde{\mathcal{Q}}\,(\tilde m)\right)\right)}{m_{k_\alpha(\delta_1)+k_\beta(\delta_1)+1}! \cdot \prod\limits_{j=1}^{k_\alpha(\delta_1)} m_j^\alpha! \prod\limits_{j=1}^{k_\beta(\delta_1)} m_j^\beta!}$$

$$\overset{(\text{ Lemma 3.8})}{\geq} -\left\{ \sum_{\widetilde{m}} \frac{m! \cdot \gamma_{k_\alpha(\delta_1)k_\beta(\delta_1)}^{m_{k_\alpha(\delta_1)+k_\beta(\delta_1)+1}} \prod_{j=1}^{k_\alpha(\delta_1)} a_j^{m_j^\alpha} \prod_{j=1}^{k_\beta(\delta_1)} b_j^{m_j^\beta}}{m_{k_\alpha(\delta_1)+k_\beta(\delta_1)+1}! \cdot \prod_{j=1}^{k_\alpha(\delta_1)} m_j^\alpha! \prod_{j=1}^{k_\beta(\delta_1)} m_j^\beta!} \right.$$

$$\left. \times \left[\sum_{j=1}^{k_\alpha(\delta_1)} m_j^\alpha \ln \alpha'_j + \sum_{j=1}^{k_\beta(\delta_1)} m_j^\beta \ln \beta'_j \right] + \mathcal{O}(\ln m) \right\}$$

$$\overset{(\text{ Lemma 3.4, (3.11)})}{\geq} -m(1-\delta_2)\left[\sum_{j=1}^{k_\alpha(\delta_1)} a_j \ln \alpha'_j + \sum_{j=1}^{k_\beta(\delta_1)} b_j \ln \beta'_j \right] + \mathrm{C}(\delta_1,\delta_2)\sqrt{m}.$$

The latter inequality is true for all sufficiently large m. □

Proposition 3.2. (*An upper bound for the entropy*) $\eta(t) = -t\ln t$, $\sum \alpha_i + \sum \beta_i = 1$, $\chi = \chi_{\alpha,\beta}^{(S)}$. *Then*

$$H_\chi\left(\vartheta_S^\chi\right) \leq \sum_j \eta(\alpha_j) + \sum_j \eta(\alpha_j).$$

Proof. First we recall the well-known construction (see [20]) of the embedding of the group $S(\infty)$ in the Powers factor. Let $n(\alpha) = \min\{i : \alpha_i > 0\}$, $n(\beta) = \min\{i : \beta_i > 0\}$, $n = n(\alpha) + n(\beta)$, $\mathfrak{N}_n^\alpha = \{1, 2, \ldots, n(\alpha)\}$, $\mathfrak{N}_n^\beta = \{-1, -2, \ldots, -n(\beta)\}$ and $\mathfrak{N}_n = \mathfrak{N}_n^\alpha \bigsqcup \mathfrak{N}_n^\beta$. We consider the algebra $M_n(\mathbb{C})$ of all complex $n\times n$-matrices with system of matrix units $\{e_{i,j}\}_{i,j\in\mathfrak{N}_n}$. Let $h_{\alpha\beta} = \operatorname{diag}\left(\alpha_1, \ldots, \alpha_{n(\alpha)}; \beta_1, \ldots, \beta_{n(\beta)}\right) \in M_n(\mathbb{C})$ and let $\varphi(\cdot) = Tr(\cdot h_{\alpha\beta})$ is the state on $M_n(\mathbb{C})$, where Tr is ordinary trace. For $j \in \mathbb{Z}$ let $\mathcal{M}_j = M_n(\mathbb{C})$ and $\varphi_j = \varphi$. Let $(\mathcal{M}, \tilde{\varphi}) = \otimes_{j\in\mathbb{Z}}(\mathcal{M}_j, \varphi_j)$, where $\tilde{\varphi} = \otimes_{j\in\mathbb{Z}}\varphi_j$. For sequence $\mathfrak{i}_k = (i_j \in \mathfrak{N}_n)_{j=-k}^{j=k}$ let

$$\mathfrak{j}(\mathfrak{i}_k) = \left(\left\{j_1 < j_2 < \ldots < j_{l(\mathfrak{i}_k)}\right\} : i_{j_l} \in \mathfrak{N}_n^\beta \quad \forall\, l = 1, 2, \ldots, l(\mathfrak{i}_k)\right).$$

If g is any permutation of the set $\mathcal{B}_k = \{-k, \ldots, 0, \ldots, k\}$, then there is permutation $s(g, \mathfrak{i}_k) \in S(g(\mathfrak{j}(\mathfrak{i}_k)))$ such that

$$s(g,\mathfrak{i}_k)(g(j_1)) < s(g,\mathfrak{i}_k)(g(j_2)) < \ldots < s(g,\mathfrak{i}_k)\left(g\left(j_{l(\mathfrak{i}_k)}\right)\right).$$

Let $\psi(g, \mathfrak{i}_k) = sgn(s(g, \mathfrak{i}_k))$ and let $\mathfrak{I}_k$ be the set of all sequences $(i_j \in \mathfrak{N}_n)_{j=-k}^{j=k}$. Now by a direct checking we can make sure, that operators $U_g = \sum\limits_{\mathfrak{i}_k\in\mathfrak{I}_k} \psi(g, \mathfrak{i}_k)\, e_{\mathfrak{i}_k\, g(\mathfrak{i}_k)}$, where

$$e_{\mathfrak{i}_k\, g(\mathfrak{i}_k)} = e_{i_{-k}\, i_{g(-k)}} \otimes e_{i_{-k+1}\, i_{g(-k+1)}} \otimes \ldots e_{i_k\, i_{g(k)}} \in \mathcal{M}_{\tilde{\varphi}},$$

$\mathcal{M}_{\tilde{\varphi}}$ is the centralizer of $\tilde{\varphi}$, define a unitary representation of the group $S(2k+1) = S(\mathcal{B}_k)$ and $\tilde{\varphi}(U_g) = \chi^{(S)}_{\alpha,\beta}(g)$. Further, we notice that automorphism ϑ^{χ}_S of the W^*-algebra generated by U_g $\left(g \in \bigcup\limits_k S(\mathcal{B}_k)\right)$ extends to the automorphism θ of the W^*-algebra $\mathcal{M}$. But it is well-known (see [5]), that θ is a noncommutative Bernouli shift with the entropy $\sum\limits_j \eta(\alpha_j) + \sum\limits_j \eta(\beta_j)$. □

Proof of Theorem 3.1. The statement of the theorem follows from Lemma 3.2 and Proposition 3.2 when sets $\{\alpha_i \neq 0\}$ and $\{\beta_i \neq 0\}$ are finite ones. In the general case we consider the algebra $\mathcal{M}_m$ that is generated by A_0 and $\pi_\chi(S(m))$ as a W^*-algebra. Using a method by which we have proved Lemma 2.1, we receive the following estimation

$$H_\chi\left(\vartheta^{\chi}_S\right) \geq \frac{H_\chi(\mathcal{M}_n)}{m}.$$

Hence, tacking into account Propositions 3.2 and 3.1, we complete the proof. □

References

[1] A. Berele, A. Regev, *Hook Young diagrams with applications to combinatories and to representations of Lie superalgebras* // Adv. Math. **64** (1987), 118–175.

[2] S. Bezuglyi, V. Ya. Golodets, *Dynamical entropy for Bogoliubov actions of free abelian groups on the CAR-algebra* // Ergod. Th. and Dynam. Sys., **17** (1997), 757–782.

[3] M. Boyko, N. Nessonov, *Entropy of Bogoliubov automorphisms on II_1-representations of the group $U(\infty)$* // Metods Funct. Anal. Topology, **9** (2003), No. 4, 317–332.

[4] A. Connes, H. Narnhofer, W. Thirring, *Dynamical entropy of C^*-algebras and von Neumann algebras* // Commun. Math. Phys., **112** (1987), 691–719.

[5] A. Connes, E. Størmer, *Entropy for automorphisms of II_1 von Neumann algebras* // Acta Math. **134** (1975), 289–306.

[6] V. Ya. Golodets, S. Neshveyev, *Entropy for Bogoliubov actions of torsion-free abelian groups on the CAR-algebra* // Ergod. Th. and Dynam. Sys., **20** (2000), No. 4, 1111–1125.

[7] V. Ya. Golodets, E. Størmer, *Entropy of C^* dynamical systems defined by bitstreams* // Ergod. Th. and Dynam. Sys., **18** (1998), 859–874.

[8] S. Kerov, A. Okounkov, G. Olshanski, *The boundary of Young graph with Jack edge multiplicities* // Intern. Math. Res. Notices (1998), No. 4, 173–199.

[9] A. N. Kolmogorov, *The new metric invariant of transitive dynamical system and automorphisms of Lebesgue spaces* // Dokl. Acad. Nauk SSSR, (1958), No. 5, 861–864. (in Russian)

[10] I. G. Macdonald, *Symmetric Functions and Hall Polinomials*. Clarendon Press·Oxford 1979.

[11] H. Narnhofer, W. Thirring, *Dynamical entropy of quantum systems and their abelian counterpart*. On Klauder Path: A field trip, Emch, G. G., Hegerfeldt, G. C., Streit, L. World Scientific; Singapore, 1994.

[12] S. Neshveyev, *Entropy of Bogoliubov automorphismes of CAR and CCR algebras with respect to quasi-free states* // Rev. Math. Phys., **13** (2001), No. 1, 29–50.

[13] A. Okounkov, *The Thoma theorem and representations of the infinite bisymmetric group* // Funct. Anal. Appl. **28** (1994), No. 2, 100–107.

[14] Y. M. Park, H. H. Shin, *Dynamical entropy of space translations of CAR and CCR algebras with respect to quasi-free states* // Commun. Math. Phys., **152** (1993), 497–537.

[15] V. A. Rokhlin, Ya. G. Sinai, *Construction and properties of invariant measurable partitions* // Dokl. Acad. Nauk SSSR, (1961), No. 5, 1038–1041. (in Russian)

[16] Ya. G. Sinai, *On the concept of the entropy for a dynamical system* // Dokl. Acad. Nauk SSSR, (1959), No. 4, 768–771. (in Russian)

[17] E. Størmer, D. Voiculescu, *Entropy of Bogoliubov automorphisms of the Canonical Anticommutation Relations* // Commun. Math. Phys., **133** (1990), 521–542.

[18] A. M. Vershik, S. V. Kerov, *Asymptotic form of minimal and typical dimensions of irreducible representations for the symmetric group* // Funk. Anal. i ego Prilozh. **19** (1985), No. 1, 25–36. (in Russian)

[19] A. M. Vershik, S. V. Kerov, *Asymptotic theory of characters of the symmetric group* // Funct. Anal. Appl. **15** (1981), No. 1, 246–255.

[20] A. M. Vershik, S. V. Kerov, *Characters and factor representations of the infinite symmetric group* // Soviet Math. Dokl., **23** (1981), No. 2, 389–392.

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