J-selfadjoint ordinary differential operators similar to selfadjoint operators*

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Abstract

Some class of J-selfadjoint ordinary differential operators is investigated. A criterion of similarity to selfadjoint operators is obtained for this class.

Consider the operator $A = (\operatorname{sgn} x) p\left(-i\frac{d}{dx}\right)$ in the Hilbert space $L^2(\mathbb{R})$, where

$$p(z) = z^{2n} + a_{2n-1}z^{2n-1} + \ldots + a_1 z + a_0 \tag{0.1}$$

is a polynomial with real coefficients a_j. Denote $D = -i\frac{d}{dx}$. Then $L = p(D)$ is a constant coefficient differential operator in $L^2(\mathbb{R})$ defined on the Sobolev space $W_2^{2n}(\mathbb{R})$ (see [1]). It is clear that $L = L^*$. Let $J = J^* = J^{-1}$ be the multiplication operator defined by $(Jf)(x) = (\operatorname{sgn} x)f(x)$, $x \in \mathbb{R}$. The operator $A = JL$, being a product of two selfadjoint noncommuting operators, is a nonselfadjoint one.

If $p(t)$ is nonnegative (i.e. $p(t) \geq 0$ for all $t \in \mathbb{R}$), then $JA = L = L^* \geq 0$ and the operator A is a definitizable operator. The resolvent set $\rho(A)$ is nonempty and therefore M. Krein-H. Langer's spectral theory can be applied. In particular, this spectral theory shows that A has a spectral function defined on open intervals in $\mathbb{R}$ with the endpoints different from 0 and ∞. The positive (negative, respectively) spectral points are of positive (negative, respectively) type. Hence 0 and ∞ are the only possible critical points. Starting with this fact B. Ćurgus and B. Najman proved in [3] that $(\operatorname{sgn} x)\frac{d^2}{dx^2}$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$. Later they extended this result to more general polynomials p. Namely, in [4] they proved, that *the operator A is similar to a selfadjoint operator if $p(t)$ is a nonnegative polynomial with at most one real root.* In the paper [5] corresponding questions for partial differential operators was studied. For more detailed references see [5]. We reproved the result of [3] in [7].

In this paper we obtain the following criterion for an operator A to be similar to a selfadjoint one:

Theorem 0.1. *Let $p(z) = z^{2n} + a_{2n-1}z^{2n-1} + \ldots + a_1 z + a_0$ be an even order polynomial with real coefficients. Then the operator $A = (\operatorname{sgn} x)p(D)$ is similar to a selfadjoint operator if and only if the polynomial $p(t)$ is nonnegative.*

Our approach is completely different. It is based on the following similarity criterion which has been obtained by S. N. Naboko [10] and M. M. Malamud [9] (see also [2], where this criterion has also been obtained under an additional assumption).

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Theorem 0.2 ([10],[9]). *A closed operator A with real spectrum acting in a Hilbert space H is similar to a selfadjoint one if and only if*

$$\varepsilon \int_{-\infty}^{+\infty} \|R_A(\mu + i\varepsilon) f\|^2 d\mu \le m_+ \|f\|^2 \qquad \forall \varepsilon > 0, \forall f \in H, \tag{0.2}$$

$$\varepsilon \int_{-\infty}^{+\infty} \|R_{A^*}(\mu + i\varepsilon) f\|^2 d\mu \le m_+^* \|f\|^2 \qquad \forall \varepsilon > 0, \forall f \in H, \tag{0.3}$$

where $R_A(\lambda) := (A - \lambda I)^{-1}$ and m_+, m_+^* are some constants independent of $f \in H$.

Thus, we have to find and estimate the resolvent of A. This problem leads to the investigation of roots of the equations $p(\xi) \pm \lambda = 0$.

1 The roots of the polynomials $p(\xi) \pm \lambda$

Let $p(z)$ be polynomial (0.1). Denote $Z_0 := \{z \in \mathbb{C} : p'(z) = 0\}$, $\Lambda_0 := p(Z_0)$, $M_0 := \Lambda_0 \cap \mathbb{R}$. The sets Z_0, Λ_0, M_0 are finite. Let $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_{2n-1}\} := \{|\operatorname{Im} \lambda_0| : \lambda_0 \in \Lambda_0 \cap \mathbb{C}_+\} \cup \{0\}$, where ε_j are arranged in ascending order, $0 = \varepsilon_0 \le \varepsilon_1 \le ... \le \varepsilon_{2n-1}$. Let $\varepsilon_{2n} := +\infty$. Denote $\Lambda_+^k := \{\lambda \in \mathbb{C} : \varepsilon_k < \operatorname{Im} \lambda < \varepsilon_{k+1}\}$, $k = 0, ..., 2n-1$. Then $\Lambda_+^j \cap \Lambda_+^k = \emptyset$ for $j \ne k$ and $\cup \overline{\Lambda_+^N} = \overline{\mathbb{C}_+}$. Here $\overline{G}$ stands for the closure of G. Let Λ_+ be one of the sets Λ_+^k, $k = 0, ..., 2n-1$.

We say that a family of functions $\{\xi_j(\lambda)\}_1^{2n}$ is a complete solution, if for fixed λ numbers $\{\xi_j(\lambda)\}_1^{2n}$ present all the roots of the equation $p(\xi) - \lambda = 0$ counting their multiplicities.

Lemma 1.1. *Let G_0 and G be simply connected domains in $\mathbb{C}$ such that $G_0 \subset G$, $G \cap \Lambda_0 = \emptyset$ and $\overline{G_0} \setminus \Lambda_0 \subset G$. Then there exists a complete solution $\{\xi_j(\lambda)\}_1^{2n}$ of the equation $p(\xi) - \lambda = 0$ such that functions $\xi_j(\lambda)$ are holomorphic in G and continuous in $\overline{G_0}$.*

Proof. Let $\lambda_0 \in \mathbb{C} \setminus \Lambda_0$ and let ξ^0 be any solution of $p(\xi) - \lambda_0 = 0$. Then $\xi^0 \notin Z_0$, $p'(\xi^0) \ne 0$ and ξ^0 is a simple root of the polynomial $p(\xi) - \lambda_0$. By the theorem on local inversion of holomorphic function there exist a neighborhood $U(\lambda_0) = \{|\lambda - \lambda_0| < r(\lambda_0)\}$ and a holomorphic function $\xi(\lambda)$ defined in $U(\lambda_0)$ such that $\xi(\lambda_0) = \xi^0$, $p(\xi(\lambda)) = \lambda$, $\xi(\lambda) \ne \xi^0$ for $\lambda \in U(\lambda_0) \setminus \{\lambda_0\}$.

Choose an arbitrary point $\lambda_1 \in G$. Since $G \cap \Lambda_0 = \emptyset$, we have $\lambda_1 \notin \Lambda_0$. Thus, the equation $p(\xi) - \lambda_1 = 0$ has $2n$ simple roots ξ_j^0, $j = 1, ..., 2n$. For each of them there exist a neighborhood $U_j(\lambda_1)$ and a holomorphic function $\xi_j(\lambda)$ defined in $U_j(\lambda_1)$ such that $\xi_j(\lambda_1) = \xi_j^0$, $p(\xi_j(\lambda)) = \lambda$. The pair $(U_j(\lambda_1), \xi_j(\lambda))$ is an analytical element. By virtue of the previous paragraph this analytical element can be continued along any curve γ in the simply connected domain G such that $p(\xi_j(\lambda)) = \lambda$. Then the monodromy theorem [12] implies that all the analytical functions $\xi_j(\lambda)$ are univalent in G. Since $p(\xi_j(\lambda)) = \lambda$ and $G \cap \Lambda_0 = \emptyset$, we have $p'(\xi_j(\lambda)) \ne 0$ for $\lambda \in G$. Hence, $\xi_j(\lambda)$ is a simple root of a polynomial $p(\xi) - \lambda$ for all $j = 1, ..., 2n$. Thus, the family $\{\xi_j(\lambda)\}_1^{2n}$ of holomorphic functions is a complete solution.

The functions $\xi_j(\lambda)$ are continuous in $\overline{G_0} \setminus \Lambda_0$ as they are holomorphic in G. Thus, it remains to show that the functions $\xi_j(\lambda)$ are continuous at the points $\lambda \in \overline{G_0} \cap \Lambda_0$. Let $\lambda_0 \in \overline{G_0} \cap \Lambda_0$ and let z_0 be a root of multiplicity k of the polynomial $p(z) - \lambda_0$. Then the theorem on local inversion of holomorphic function implies that for all $r > 0$ there exists $\mu_0(r) > 0$ such that for all $\lambda \in U(\lambda_0) := \{|\lambda - \lambda_0| < \mu_0\}$ the polynomial $p(\xi)$ takes on value λ in exactly k distinct points of the circle $|\xi - z_0| < r$. If $\lambda \in G \cap U(\lambda_0)$, these points belong to the set $\{\xi_j(\lambda)\}_1^{2n}$.

Denote them by $\xi_{jq}(\lambda)$, $q=1,...,k$. Define $\xi_{jq}(\lambda_0):=z_0$. If $r\to 0$ then $p(\xi)=\lambda\to\lambda_0$ and $\lim\limits_{\lambda\to\lambda_0}\xi_{jq}(\lambda)=z_0$. So we have continuously prolonged the functions $\xi_{jq}(\lambda)$ to the point λ_0. The set $\overline{G_0}\cap\Lambda_0$ is finite. Arguing similarly for all the roots of $p(z)-\lambda_0$ and for all $\lambda_0\in\overline{G_0}\cap\Lambda_0$ we continuously prolong the functions $\xi_j(\lambda)$ on $\overline{G_0}$. □

Applying Lemma 1.1 to the domain $G_0=\Lambda_+$, we conclude that the equations $p(\xi)-\lambda=0$ and $p(\xi)+\lambda=0$ have complete solutions $\Xi^+=\left\{\xi_j^+(\lambda)\right\}_1^{2n}$ and $\Xi^-=\left\{\xi_j^-(\lambda)\right\}_1^{2n}$, respectively, defined in Λ_+.

By $\omega_{m,j}(z)$ we denote the branch of the multifunction $\sqrt[m]{z}$ defined in $\mathbb{C}$ with a cut along the positive semiaxis $\mathbb{R}_+$ and fixed by $\omega_{m,j}(-1)=e^{i\left(\frac{\pi}{m}+\frac{2\pi(j-1)}{m}\right)}$.

Lemma 1.2. *There exists a numbering of functions $\xi_j^+(\lambda)$, $\xi_j^-(\lambda)$ such that*

$$\frac{\xi_j^+(\lambda)}{\omega_{2n,j}(\lambda)}\to 1,\quad \frac{\xi_j^-(\lambda)}{\omega_{2n,2n-j+1}(-\lambda)}\to 1$$

as $\lambda\to\infty$, $\lambda\in\Lambda_+$, $j=1,\dots,2n$.

Proof. Clearly, 0 is a zero of multiplicity $2n$ of the function $\phi(\zeta):=\frac{1}{p\left(\frac{1}{\zeta}\right)}$. Therefore $\phi(\zeta)=\phi_1(\zeta)\zeta^{2n}$, where $\phi_1(\zeta)=\frac{\phi(\zeta)}{\zeta^{2n}}=\frac{\frac{1}{\zeta^{2n}}}{p\left(\frac{1}{\zeta}\right)}$ is holomorphic in a neighborhood of 0 and $\phi_1(0)=1$. Let $U(0)$ be a neighborhood of 0 where $\phi_1(\zeta)\neq 0$. Let $\psi_1(\zeta)$ be the branch of the multifunction $\sqrt[2n]{\phi_1(\zeta)}$ fixed by $\psi_1(0)=1$. The only zero of the function $\psi(\zeta):=\psi_1(\zeta)\,\zeta$ in $U(0)$ is the simple zero 0. If $p\left(\frac{1}{\zeta}\right)=\lambda$, $\phi(\zeta):=\psi^{2n}(\zeta)=\lambda^{-1}$. By the theorem on local inversion of a holomorphic function for each $j=1,...,2n$ and for all λ sufficiently large there exists the only function $\zeta_q(\lambda)$ taking values in $U(0)$ such that $\psi_1\left(\zeta_j(\lambda)\right)\zeta_j(\lambda)=\frac{1}{\omega_{2n,j}(\lambda)}$. Then $\zeta_j(\lambda)\to 0$ as $\lambda\to\infty$ and

$$\phi_1\left(\zeta_j(\lambda)\right)\zeta_j^{2n}(\lambda)=\phi\left(\zeta_j(\lambda)\right)=\frac{1}{p\left(\frac{1}{\zeta_j(\lambda)}\right)}=\frac{1}{\lambda},$$

i.e., $p\left(\frac{1}{\zeta_j(\lambda)}\right)-\lambda=0$. Consequently, there exists a numbering of $\xi_j^+(\lambda)$ such that $\xi_j^+(\lambda)=\frac{1}{\zeta_j(\lambda)}$, $j=1,...,2n$. But

$$\frac{\xi_j^+(\lambda)}{\omega_{2n,j}(\lambda)}=\frac{1}{\zeta_j(\lambda)\omega_{2n,j}(\lambda)}=\psi_1\left(\zeta_j(\lambda)\right)\to 1$$

as $\lambda\to\infty$. The proof for $\xi_q^-(\lambda)$ is the same. □

Assume that the functions $\xi_j^\pm$ are enumerated as in the statement of Lemma 1.2. Now we can introduce the following families of functions: $\Xi:=\Xi^+\cup\Xi^-$, $\Xi_1^+=\{\xi_j^+\}_1^n$, $\Xi_2^+=\{\xi_j^+\}_{n+1}^{2n}$, $\Xi_1^-=\{\xi_j^-\}_1^n$, $\Xi_2^-=\{\xi_j^-\}_{n+1}^{2n}$. For an arbitrary family $\Phi\subset\Xi$, $\Phi[z_0,\lambda_0]$ denotes the family of all functions $\xi(\lambda)$ defined on $\overline{\Lambda_+}$ such that $\xi\in\Phi$ and $\xi(\lambda_0)=z_0$. Let $\sharp(\Phi)$ be the number of elements in a family Φ.

Let $\xi\in\Xi$, $\lambda\in\Lambda_+$. Then, by virtue of Lemma 1.1, $\xi(\lambda)$ is a continuous in Λ_+ function. Obviously, $\xi(\lambda)\notin\mathbb{R}$ for all $\lambda\in\Lambda_+$. Therefore, either $\xi(\lambda)\in\mathbb{C}_+$ for all $\lambda\in\Lambda_+$ or $\xi(\lambda)\in\mathbb{C}_-$ for all $\lambda\in\Lambda_+$. The polynomial equation $z^{2n}-\lambda=0$ has exactly n solutions $\omega_{2n,j}(\lambda)$, $j=1,...,n$ in $\mathbb{C}_+$ and exactly n solutions $\omega_{2n,j}(\lambda)$, $j=n+1,...,2n$ in $\mathbb{C}_-$. Analogously, the equation $z^{2n}+\lambda=0$ has exactly n solutions $\omega_{2n,j}(-\lambda)$, $j=1,...,n$ in $\mathbb{C}_-$ and exactly n solutions $\omega_{2n,j}(-\lambda)$, $j=n+1,...,2n$ in $\mathbb{C}_+$. Therefore one derives from Lemma 1.2 the following lemma.

Lemma 1.3. *Each of the families Ξ^+, Ξ^- contains exactly n functions taking values in $\mathbb{C}_+$ and exactly n functions taking values in $\mathbb{C}_-$. Namely, the functions from Ξ_1^+,Ξ_2^- take values in $\mathbb{C}_+$, the functions from Ξ_2^+,Ξ_1^- take values in $\mathbb{C}_-$.*

If $\lambda_0 \in \Lambda_+$, then all the roots of polynomials $p(z) \mp \lambda_0$ are simple. If $\lambda_0 \in \overline{\Lambda_+}$ and z_0 is a root of multiplicity k of the polynomial $p(z) - \lambda_0$ (respectively $p(z) + \lambda_0$), then by Lemma 1.1 $\sharp(\Xi^+[z_0, \lambda_0]) = k$ (respectively $\sharp(\Xi^-[z_0, \lambda_0]) = k$).

Lemma 1.4. *Let $\lambda_0 \in \overline{\Lambda_+}$ and let z_0 be a root of multiplicity k of the polynomial $p(z) - \lambda_0$ (respectively $p(z) + \lambda_0$). Let $\Xi^+[z_0, \lambda_0] = \{\tilde{\xi}_q^+(\lambda)\}_1^k$ (respectively $\Xi^-[z_0, \lambda_0] = \{\tilde{\xi}_q^-(\lambda)\}_1^k$). Then for $q = 1, \ldots, k$ and for $\lambda \in \Lambda_+$ sufficiently close to λ_0*

$$\tilde{\xi}_q^+(\lambda) - z_0 = \gamma\left(\tilde{\xi}_q^+(\lambda)\right)\omega_{k,q}(\lambda - \lambda_0)$$
$$\left(\textit{respectively } \tilde{\xi}_q^-(\lambda) - z_0 = \gamma\left(\tilde{\xi}_q^-(\lambda)\right)\omega_{k,q}(-\lambda + \lambda_0)\right).$$

Here $\gamma(z)$ is an arbitrary holomorphic in a neighborhood of z_0 branch of the multifunction $\frac{1}{\sqrt[k]{p_1(z)}}$, where $p_1(z) = \dfrac{p(z) - \lambda_0}{(z - z_0)^k}$ (respectively $p_1(z) = \dfrac{p(z) + \lambda_0}{(z - z_0)^k}$).

Proof. By the conditions of the lemma we have $p(z) - \lambda_0 = (z - z_0)^k p_1(z)$, $p_1(z_0) \neq 0$. Let $U(z_0)$ be a neighbourhood of z_0 such that $p_1(z) \neq 0$ for $z \in U(z_0)$. Let us choose a branch of the multifunction $\sqrt[k]{p_1(z)}$ in $U(z_0)$. In $U(z_0)$ the function $\phi(z) := \sqrt[k]{p_1(z)}(z - z_0)$ has the only root z_0, which is simple. Hence, for λ sufficiently close to λ_0 and for each $q = 1, ..., k$ there exists the sole family of functions $z_q(\lambda)$ taking values in $U(z_0)$ with the property $\sqrt[k]{p_1(z_q)}(z_q - z_0) = \omega_{k,q}(\lambda - \lambda_0)$. Indeed, $p_1(z_q)(z_q - z_0)^k = \lambda - \lambda_0$ and the last statement follows from the theorem on local inversion of a holomorphic function. Consequently, $p(z_k) - \lambda = 0$ and we can enumerate $\xi_q^+(\lambda)$ so that $\tilde{\xi}_q^+(\lambda) = z_q(\lambda)$. Then

$$\tilde{\xi}_q^+(\lambda) = \frac{1}{\sqrt[k]{p_1\left(\tilde{\xi}_q^+(\lambda)\right)}}\omega_{k,q}(\lambda - \lambda_0)$$

for λ sufficiently close to λ_0.

The proof for $\xi_q^-(\lambda)$ is completely similar. □

Let $k(z_0, \lambda_0)$ be the multiplicity of a root z_0 of the polynomial $p(z) - \lambda_0$. Lemma 1.1 implies that $\sharp(\Xi_1^\pm[z_0, \lambda_0]) + \sharp(\Xi_2^\pm[z_0, \lambda_0]) = k(z_0, \pm\lambda_0)$. Assume that $z_0 \in \mathbb{C}_+$. Then $\sharp(\Xi_2^+[z_0, \lambda_0]) = \sharp(\Xi_1^-[z_0, \lambda_0]) = 0$ and, therefore

$$\begin{aligned}\sharp(\Xi_1^+[z_0, \lambda_0]) = \sharp(\Xi^+[z_0, \lambda_0]) = k(z_0, \lambda_0)\ ,\\ \sharp(\Xi_2^-[z_0, \lambda_0]) = \sharp(\Xi^-[z_0, \lambda_0]) = k(z_0, -\lambda_0)\ .\end{aligned} \tag{1.1}$$

Analogously, if $z_0 \in \mathbb{C}_+$, then

$$\begin{aligned}\sharp(\Xi_2^+[z_0, \lambda_0]) = \sharp(\Xi^+[z_0, \lambda_0]) = k(z_0, \lambda_0)\ ,\\ \sharp(\Xi_1^-[z_0, \lambda_0]) = \sharp(\Xi^-[z_0, \lambda_0]) = k(z_0, -\lambda_0)\ .\end{aligned} \tag{1.2}$$

Let $\xi^+(\lambda) \in \Xi^+$, then $p(\xi^+(\lambda)) - \lambda = 0$ and $p\left(\overline{\xi^+(\lambda)}\right) + (-\bar{\lambda}) = 0$. It is equivalent to $\xi^-(\lambda) := \overline{\xi^+(-\bar{\lambda})} \in \Xi^-$. Taking into account Lemma 1.2, we see that $\xi_j^-(\lambda) = \overline{\xi_j^+(-\bar{\lambda})}$, $j = 1, ..., 2n$, and therefore one arrives at once

$$\sharp(\Xi_1^+[\overline{z_0}, -\overline{\lambda_0}]) = \sharp(\Xi_1^-[z_0, \lambda_0]), \quad \sharp(\Xi_2^+[\overline{z_0}, -\overline{\lambda_0}]) = \sharp(\Xi_2^-[z_0, \lambda_0]). \tag{1.3}$$

If $z_0 \in \mathbb{R}$ is a root of one of the polynomials $p(z) \mp \lambda_0$, then $\lambda_0 \in \mathbb{R}$. In this case the distribution of roots among $\mathbb{C}_+$ and $\mathbb{C}_-$ is provided by the following lemma.

Lemma 1.5. *Let* $\Lambda_+ = \Lambda_0^+$. *Let* $\mu_0 \in \mathbb{R} \subset \overline{\Lambda_+}$, $t_0 \in \mathbb{R}$.

(a) If $k(t_0,\mu_0)$ *is an even number, then*

$$\sharp(\Xi_1^+[t_0,\mu_0]) = \sharp(\Xi_2^+[t_0,\mu_0]) = \sharp(\Xi_1^-[t_0,-\mu_0]) = \\ = \sharp(\Xi_2^-[t_0,-\mu_0]) = \frac{1}{2}k(t_0,\mu_0). \quad (1.4)$$

(b) If $k(t_0,\mu_0) = 2k_1 + 1$ *is an odd number, then one of the following assertions is valid*

$$(i) \quad \begin{aligned} \sharp(\Xi_1^+[t_0,\mu_0]) &= \sharp(\Xi_1^-[t_0,-\mu_0]) = k_1, \\ \sharp(\Xi_2^+[t_0,\mu_0]) &= \sharp(\Xi_2^-[t_0,-\mu_0]) = k_1 + 1. \end{aligned} \quad (1.5)$$

$$(ii) \quad \begin{aligned} \sharp(\Xi_1^+[t_0,\mu_0]) &= \sharp(\Xi_1^-[t_0,-\mu_0]) = k_1 + 1, \\ \sharp(\Xi_2^+[t_0,\mu_0]) &= \sharp(\Xi_2^-[t_0,-\mu_0]) = k_1. \end{aligned} \quad (1.6)$$

Proof. We set $k = k(t_0,\mu_0)$. In the case $k \le 1$ the lemma is trivial. Let $\Xi^+[t_0,\mu_0] = \{\tilde{\xi}_q^+(\lambda)\}_1^k$, where the functions $\tilde{\xi}_q^+(\lambda)$ are enumerated according to Lemma 1.4. Since $t_0 \in \mathbb{R}$ and $\mu_0 \in \mathbb{R}$, the polynomial $p_1(z) = \frac{p(z)-\mu_0}{(z-t_0)^k}$ has real coefficients and, therefore $p_1(t_0) = p_1\left(\tilde{\xi}_q^+(\mu_0)\right) \in \mathbb{R} \setminus \{0\}$. Choose $\gamma(z)$ such that $\arg\gamma(z_0) = 0$ in the case $p_1(z_0) > 0$, and $\arg\gamma(z_0) = \frac{\pi}{k}$ in the case $p_1(z_0) < 0$. In what follows we consider all angles modulo 2π. Note that if $\lambda = \mu_0 + i\varepsilon$, where $\varepsilon > 0$ is small enough, then

$$\arg\tilde{\xi}_q^+(\lambda) = \arg\omega_{k,q}(\lambda - \lambda_0) + \arg\gamma\left(\tilde{\xi}_q^+(\lambda)\right),$$
$$\arg\omega_{k,q}(\lambda - \lambda_0) = \frac{\pi}{2k} + \frac{2\pi}{k}(q-1),$$

and $\arg\gamma\left(\tilde{\xi}_q^+(\lambda)\right) \to \arg\gamma(z_0)$ as $\varepsilon \to 0$.

Assume that $k = 2k_1 \ge 2$ is an even number. Then for any $p_1(t_0)$

$$\forall\alpha > 0\ \exists\delta_\varepsilon : \varepsilon < \delta_\varepsilon \Longrightarrow -\alpha < \arg\gamma\left(\tilde{\xi}_q^+(\lambda)\right) < \frac{\pi}{k} + \alpha,$$

and, therefore

$$\frac{\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < \pi - \frac{\pi}{2k} + \alpha, \quad q = 1, ..., k_1,$$
$$\pi + \frac{\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < 2\pi - \frac{\pi}{2k} + \alpha, \quad q = k_1 + 1, ..., 2k_1.$$

Thus $\tilde{\xi}_q^+(\lambda) \in \mathbb{C}_+$ for $q = 1, ..., k_1$, $\tilde{\xi}_q^+(\lambda) \in \mathbb{C}_-$ for $q = k_1 + 1, ..., 2k_1$. This proves (a).

Assume that $k = 2k_1 + 1 \ge 3$ is an odd number. Then in the case $p_1(t_0) > 0$ we have

$$\forall\alpha > 0\exists\delta_\varepsilon : \varepsilon < \delta_\varepsilon \Longrightarrow \frac{\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < \pi - \frac{\pi}{2k} + \alpha,\ q = 1, ..., k_1 + 1,$$
$$\pi + \frac{3\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < 2\pi - \frac{3\pi}{2k} + \alpha,\ q = k_1 + 2, ..., 2k_1 + 1,$$

and, therefore assertion (ii) holds. In the case $p_1(t_0) < 0$ we have

$$\forall\alpha > 0\,\exists\delta_\varepsilon : \varepsilon < \delta_\varepsilon \Longrightarrow \frac{3\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < \pi - \frac{3\pi}{2k} + \alpha,\ q = 1, ..., k_1,$$
$$\pi + \frac{\pi}{2k} - \alpha < \arg\tilde{\xi}_q^+(\lambda) < 2\pi - \frac{\pi}{2k} + \alpha,\ q = k_1+1, ..., 2k_1+1,$$

and, therefore assertion (i) holds.

The statements for $\Xi^-[t_0, -\mu_0] = \left\{\tilde{\xi}_j^-(\lambda)\right\}_1^k$ follow from (1.3). □

We need Lemma 1.6 to estimate some integrals.

Proposition 1.1. *Let $z = \eta(t)$ be a rectifiable curve lying in the circle $U = \{z : |z - z_0| < R\}$. Assume that $\sup\limits_{t_1,t_2\in T} |\arg\eta'(t_1) - \arg\eta'(t_2)| < \pi/2$ for some set T such that $[a, b] \setminus T$ is a set of the zero Lebesgue measure. Then the length of the curve $z = \eta(t)$ is less then $4R$, that is $\int\limits_a^b |\eta'(t)|dt < 4R$.*

Proof. By conditions we can choose $z_1 \in \mathbb{C}$ such that

$$\sup_{t\in T} |\arg\eta'(t) - \arg z_1| < \pi/4. \tag{1.7}$$

Let $l_1 := \{\alpha z_1 : \alpha \in \mathbb{R}\}$. Denote by $z = \tilde{\eta}(t)$ the orthogonal projection of the curve $z = \eta(t)$ on the line l_1. Since $\eta(t) \in U$ for all $t \in [a, b]$, the length of the curve $z = \tilde{\eta}(t)$ is less then 2R, $\int\limits_a^b |\tilde{\eta}'(t)|dt \le 2R$. On the other hand, by virtue of (1.7), we have

$$\int_a^b |\tilde{\eta}'(t)|dt > \int_a^b \frac{1}{2}|\eta'(t)|dt.$$

The combination of these inequalities proves the proposition. □

Lemma 1.6. *Let $\lambda = \mu + i\varepsilon$, $\operatorname{Im}\lambda = \varepsilon$, $\xi \in \Xi$, $\xi(\lambda) \in \mathbb{C}_\pm$ for all $\lambda \in \Lambda_+$. Then there exists a constant M_C such that for all $\varepsilon > 0$ and for any function $F(z)$ from the Hardy space $H^2(\mathbb{C}_\pm)$ the following inequality is valid:*

$$\int_{-\infty}^{+\infty} |F(\xi(\lambda))|^2 |d\xi(\lambda)| := \int_{-\infty}^{+\infty} |F(\xi(\lambda))|^2 |\xi'(\lambda)|d\mu \le M_C\|F\|^2_{H^2}. \tag{1.8}$$

(Henceforth we take integrals along lines $\lambda = \mu + i\varepsilon$, where ε is fixed.)

Proof. We establish inequality (1.8) for $\xi \in \Xi_1^+$. According to the well known Carleson embedding theorem (see, for example, [8, viii E], [6],[11]) it is sufficient to prove that for all $\varepsilon > 0$, $R > 0$, $z_0 \in \mathbb{R}$ there exists a constant M such that

$$\int_{|\xi(\lambda)-z_0|<R} |\xi'(\lambda)|\, d\mu \le MR, \tag{1.9}$$

that is, the length of the part of the curve

$$\{z = \xi(\lambda), \lambda = \mu + i\varepsilon, \mu \in (-\infty, +\infty)\} \cap \{|z - z_0| < R\}$$

is less then MR.

By Lemmas 1.2 and 1.4 we obtain that for all $\lambda_0 \in \overline{\mathbb{C}_+} \cup \{\infty\}$ there exist a neighborhood $U(\lambda_0)$ of λ_0, a number $k(\lambda_0) \in \{1, ..., 2n\}$ and a holomorphic in a neighborhood of the point $\xi(\lambda_0)$ function $\gamma_{\lambda_0}(z)$ such that $\gamma_{\lambda_0}(\xi(\lambda_0)) \neq 0$ and one of the following equalities is valid for $\lambda \in U(\lambda_0)$

$$\begin{aligned} &\xi(\lambda) = \xi(\lambda_0) + \gamma_{\lambda_0}(\xi(\lambda))\, \omega_{k(\lambda_0),1}(\lambda - \lambda_0) && \text{for } \lambda_0 \neq \infty, \\ &\xi(\lambda) = \gamma_\infty(\xi(\lambda))\, \omega_{2n,1}(\lambda) && \text{for } \lambda_0 = \infty. \end{aligned}$$

(If $\lambda_0 = \infty$ we set for convenience $\omega_{k(\lambda_0),1}(\lambda - \lambda_0) := \omega_{2n,1}(\lambda)$.) Now one derives

$$\xi'(\lambda) = \frac{\gamma_{\lambda_0}(\xi(\lambda))}{1 - \gamma'_{\lambda_0}(\xi(\lambda))\, \omega_{k(\lambda_0),1}(\lambda - \lambda_0)} \left(\omega_{k(\lambda_0),1}(\lambda - \lambda_0)\right)'.$$

Therefore,

$$|\xi'(\lambda)| \leq C(\lambda_0) \left| \left(\omega_{k(\lambda_0),1}(\lambda - \lambda_0)\right)' \right| \tag{1.10}$$

for λ sufficiently close to λ_0. Constricting $U(\lambda_0)$ we suppose that the last inequality is valid there.

Since the set $\overline{\mathbb{C}_+} \cup \{\infty\}$ is compact with respect to the topology of $\overline{\mathbb{C}}$, we can extract a finite subcovering $\bigcup_{j \in I} U(\lambda_j)$ from the covering $\bigcup_{\lambda_0} U(\lambda_0)$. By virtue of (1.10) the measure $|\xi'(\lambda)|\, d\mu$ is less than or equals the measure $C(\lambda_0) \left| \left(\omega_{k(\lambda_0),1}(\lambda - \lambda_0)\right)' \right| d\mu$ on $U(\lambda_0)$. Therefore, it suffices to prove inequality (1.9) for $\xi(\lambda) = \xi_k(\lambda) = \omega_{k,1}(\lambda)$, $k = 1, ..., 2n$. For these curves, by Proposition 1.1, we have

$$\int\limits_{\substack{|\xi(\lambda) - z_0| < R \\ \mu < 0}} |\xi'(\lambda)|\, d\mu < 4R, \qquad \int\limits_{\substack{|\xi(\lambda) - z_0| < R \\ \mu > 0}} |\xi'(\lambda)|\, d\mu < 4R,$$

so inequality (1.9) holds true for $M = 8$.

The proof for other functions from Ξ is the same. Choosing the greatest constant we obtain M_C. □

2 The resolvents of the operators A, A^*

Since the operator $L = p(D) = L^*$ is closed and J is unitary, $A = JL$ is closed too. Let us define the restriction $A_0 = A\,|_{D(A_0)}$, where

$$D(A_0) = \big\{ y(x) \in W_2^{2n}(\mathbb{R}_-) \oplus W_2^{2n}(\mathbb{R}_+) : \\ (D^j y)(-0) = (D^j y)(+0) = 0,\ j = 0, ..., 2n-1 \big\}.$$

A_0 is a symmetric operator. The adjoint operator A_0^* has the domain $D(A_0^*) = W_2^{2n}(\mathbb{R}_-) \oplus W_2^{2n}(\mathbb{R}_+)$ and is defined by the same differential expression. Lemma 1.3 implies that any $\lambda \notin \mathbb{R}$ is an eigenvalue of A_0^* of multiplicity $2n$. If, in addition, $\lambda \in \mathbb{C}_+ \setminus \Lambda_0$, the functions $e^{i\xi_j^+(\lambda)x}\chi_+(x)$, $e^{i\xi_j^-(\lambda)x}\chi_-(x)$, $j = 1, ..., n$ form a basis of $\ker(A_0^* - \lambda I)$. If $\lambda \in \mathbb{C}_- \setminus \Lambda_0$, then $e^{i\xi_j^-(-\lambda)x}\chi_+(x)$,

$e^{i\xi_j^+(-\lambda)x}\chi_-(x)$, $j=n+1,...,2n$ form a basis of $\ker(A_0^*-\lambda I)$. Here $\chi_-(x)$, $\chi_+(x)$ stands for the indicator functions of $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. Thus the deficiency index of A_0 is $(2n,2n)$. Let $\tilde{A}=A_0^*\mid_{D(\tilde{A})}$, where

$$D(\tilde{A})=\big\{y(x)\in W_2^{2n}(\mathbb{R}_-)\oplus W_2^{2n}(\mathbb{R}_+): \\ (D^jy)(-0)=(D^jy)(+0)=0,\ j=0,...,n-1\big\}. \quad (2.1)$$

Then $\tilde{A}$ is a selfadjoint extension of A_0. The operators A and A^* are proper nonselfadjoint extensions of A_0 with the domains

$$D(A)=\big\{y(x)\in W_2^{2n}(\mathbb{R}_-)\oplus W_2^{2n}(\mathbb{R}_+): \\ (D^jy)(-0)=(D^jy)(+0),\ j=0,...,2n-1\big\}, \quad (2.2)$$

$$D(A^*)=\big\{y(x)\in W_2^{2n}(\mathbb{R}_-)\oplus W_2^{2n}(\mathbb{R}_+): \\ (D^jy)(-0)=-(D^jy)(+0),\ j=0,...,2n-1\big\}. \quad (2.3)$$

Clearly, $A_0\subset A\subset A_0^*$, $A_0\subset A^*\subset A_0^*$.

The following lemma is known [4, Theorem 2.2]. We present the proof for the sake of completeness.

Lemma 2.1 ([4]). *The operators A, A^* have the real spectrums, $\sigma(A)\subset\mathbb{R}$, $\sigma(A^*)\subset\mathbb{R}$.*

Proof. Let $\lambda\notin\mathbb{R}$. By Lemma 1.3 the polynomial equation $p(z)-\lambda=0$ has exactly n roots $\{z_j^+\}_1^n$ in $\mathbb{C}_+$ (counting their multiplicities). These roots lead to the standard system of linearly independent solutions $\{\phi_j^+\}_1^n$ of the homogenous equation $p(D)y-\lambda y=0$ such that $\phi_j^+(x)\chi_+(x)\in W_2^{2n}(\mathbb{R}_+)$ are eigenfunctions of A_0^*. Analogously, the equation $p(z)+\lambda=0$ has exactly n roots $\{z_j^-\}_1^n$ in $\mathbb{C}_-$. There exists the corresponding system $\{\phi_j^-\}_1^n$ of solutions of $p(D)y+\lambda y=0$ such that $\phi_j^-(x)\chi_-(x)\in W_2^{2n}(\mathbb{R}_-)$ are eigenfunctions of A_0^*. Since $z_j^+\neq z_q^-$ for all j,q, the system $\{\phi_j^+,\phi_j^-\}_1^n$ is linearly independent. Moreover, it is a basis of solutions of the homogenous equation $q(D)y=0$, where $q(t)=\prod_{j=1}^n(t-z_j^+)(t-z_j^-)$. Therefore its Wronskian does not have zeros. The functions $\phi_j^+(x)\chi_+(x)$, $\phi_j^-(x)\chi_-(x)$, $j=1,...,n$ form a basis of $\ker(A_0^*-\lambda I)$.

Since the operator $\tilde{A}$ is selfadjoint, we have $\sigma(\tilde{A})\subset\mathbb{R}$ and therefore for any $f(x)\in L^2(\mathbb{R})$ there exists the only function $\tilde{y}(x)\in D(\tilde{A})$ such that $\tilde{A}\tilde{y}-\lambda\tilde{y}=f$. Then

$$A_0^*y-\lambda y=f \quad\Longleftrightarrow\quad y(x)=\tilde{y}(x)+\sum_{j=1}^n c_j^+\phi_j^+ +\sum_{j=1}^n c_j^-\phi_j^-,$$

where $c_j^\pm$ are arbitrary numbers. Suppose that $y\in D(A)$. Then the coefficients $c_j^\pm$ clearly satisfy the system

$$\sum_{j=1}^n c_j^+\left(D^q\phi_j^+\right)(+0)+\left(D^q\tilde{y}\right)(+0)=\sum_{j=1}^n c_j^-\left(D^q\phi_j^-\right)(-0)+\left(D^q\tilde{y}\right)(-0),$$

$q=0,...,2n-1$. It follows from the continuity of the functions $\phi_j^\pm$ and its derivatives that the determinant of the system is the Wronskian of the functions $\{\phi_j^+,\phi_j^-\}_1^n$ evaluated at 0. Hence, the determinant differs from 0. Therefore, for any $f(x)\in L^2(\mathbb{R})$ there exists the only $y\in D(A)$ such that $Ay-\lambda y=f$, i. e., $\lambda\notin\sigma(A)$.

Thus, we obtain the inclusion $\sigma(A)\subset\mathbb{R}$. The identity $\sigma(A^*)=\overline{\sigma(A)}$ implies $\sigma(A^*)\subset\mathbb{R}$. □

Setting $w = \{w_j\}_1^m$, we put

$$M(w) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ w_1 & w_2 & \dots & w_m \\ \dots & \dots & \dots & \dots \\ w_1^{m-1} & w_2^{m-1} & \dots & w_m^{m-1} \end{pmatrix}.$$

Denote by $V(w)$ the Vandermonde determinant, $V(w) := \det M(w) = \prod_{1\le q<j\le m} (w_j - w_q)$. The minor of $V(w)$ obtained by excluding the j-th column and the last row is denoted by $V_j(w)$. We denote the vector-column $(w_j)_1^m$ by $\mathrm{col}(w_j)_1^m$.

Let $\lambda \in \Lambda_+$. Then the system of functions $e^{i\xi_j^+(\lambda)x}\chi_+(x)$, $e^{i\xi_j^-(\lambda)x}\chi_-(x)$, $j = 1, ..., n$ form a basis in $\ker(A_0^* - \lambda I)$. Let us introduce the family of functions

$$\Theta = \{\theta_j(\lambda)\}_1^{2n} := \{\xi_1^+(\lambda), \dots, \xi_n^+(\lambda), \xi_1^-(\lambda), \dots, \xi_n^-(\lambda)\} = \Xi_1^+ \cup \Xi_1^- \ .$$

For a family of functions $\Phi = \{\phi_j(\bullet)\}_{j\in I}$ and for fixed λ, $\Phi(\lambda)$ denotes the family of numbers $\{\phi_j(\lambda)\}_{j\in I}$. We shall sometimes abbreviate $M(\Phi(\lambda))$, $V(\Phi(\lambda))$ as $M(\Phi)$, $V(\Phi)$.

As usual we denote the Fourier transform of $f(x) \in L^2(\mathbb{R})$ by

$$\widehat{f}(x) = \mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}} \ \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{+N} e^{-ixt} f(t)\ dt \ .$$

Here l.i.m. means limit in quadratic mean. We also write $\widehat{f}(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-izt} f(t)\ dt$ for $z \in \mathbb{C}$, if the integral converges. Let $f_\pm(x) := f(x)\chi_\pm(x)$. Then $\widehat{f_+}(z)$ is in the Hardy space $H^2(\mathbb{C}_-)$, $\widehat{f_-}(z) \in H^2(\mathbb{C}_+)$. If $u(z) \in H^2(\mathbb{C}_\pm)$ and $u(t) = u(z) \mid_{\mathbb{R}}$, we shall write $u(t) \in H^2(\mathbb{C}_\pm)$. Let $\mathcal{H}f(x) := \dfrac{1}{\pi}$ v.p.-$\displaystyle\int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\ dt$ be the Hilbert transform of $f(t) \in L^2(\mathbb{R})$, where v.p.-$\int$ means the principal value integral.

It is known ([8, vi. D]) that

$$f(t) \in H^2(\mathbb{C}_+) \Rightarrow \mathcal{H}f(t) = -if(t), \quad f(t) \in H^2(\mathbb{C}_-) \Rightarrow \mathcal{H}f(t) = if(t).$$

According to Paley-Wiener theorem (see, for example, [13, vi. 4]) $\|\widehat{f_+}(z)\|_{H^2} = \|f_+(x)\|_{L^2}$, $\|\widehat{f_-}(z)\|_{H^2} = \|f_-(x)\|_{L^2}$.

At first, we find the resolvent for the function $e^{i\alpha x}\chi_+(x)$. Let $\lambda \in \Lambda_+$, $f_\alpha(x) = e^{i\alpha x}\chi_+(x)$, where $\alpha \in \mathbb{C}_+$ and $\alpha \ne \theta_j(\lambda)$ for all $j \in \{1, ..., 2n\}$. Evidently, $f_\alpha(x) \in L^2(\mathbb{R})$. Since $A_0^* f_\alpha = p(\alpha) f_\alpha$, then $y_1(x) = \frac{1}{p(\alpha)-\lambda} f_\alpha(x) \in W_2^{2n}(\mathbb{R}_+)$ is a particular solution of the equation $(A_0^* - \lambda I)y = f_\alpha$. Then the general solution is

$$y(x) = y_1(x) + \sum_{j=1}^n c_j^+ e^{i\xi_j^+ x}\chi_+(x) + \sum_{j=1}^n c_j^- e^{i\xi_j^- x}\chi_-(x).$$

In order to find $y_\alpha(x) = R_A(\lambda) f_\alpha$, we have to use condition (2.2). Taking equalities

$$(D^q y)(-0) = \sum_{j=1}^n c_j^- D^q \left(e^{i\xi_j^- x}\chi_-(x) \right)_{x=-0} = \sum_{j=1}^n c_j^- (\xi_j^-)^q,$$

$$(D^q y)(+0) = \frac{1}{p(\alpha) - \lambda}\ \alpha^q + \sum_{j=1}^n c_j^+ (\xi_j^+)^q,$$

into account one arrives at the system

$$\sum_{j=1}^{n}(-c_j^+)(\xi_j^+)^q+\sum_{j=1}^{n}c_j^-(\xi_j^-)^q=\frac{1}{p(\alpha)-\lambda}\,\alpha^q\ ,\quad q=0,...,2n-1.$$

We rewrite the system as

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1\\ \xi_1^+ & \dots & \xi_n^+ & \xi_1^- & \dots & \xi_n^-\\ (\xi_1^+)^2 & \dots & (\xi_n^+)^2 & (\xi_1^-)^2 & \dots & (\xi_n^-)^2\\ \dots & \dots & \dots & \dots & \dots & \dots\\ (\xi_1^+)^{2n-1} & \dots & (\xi_n^+)^{2n-1} & (\xi_1^-)^{2n-1} & \dots & (\xi_n^-)^{2n-1} \end{pmatrix}\begin{pmatrix} -c_1^+\\ \dots\\ -c_n^+\\ c_1^-\\ \dots\\ c_n^- \end{pmatrix}=$$

$$=M\left(\Theta(\lambda)\right)\begin{pmatrix} -c_1^+\\ \dots\\ -c_n^+\\ c_1^-\\ \dots\\ c_n^- \end{pmatrix}=\frac{1}{p(\alpha)-\lambda}\begin{pmatrix} 1\\ \alpha\\ \alpha^2\\ \dots\\ \alpha^{2n-1} \end{pmatrix}.$$

Since all the numbers $\xi_j^\pm$, $j=1,...,n$ are different, the determinant of the system is $V(\Theta)=V(\xi_1^+,...,\xi_n^+,\xi_1^-,...,\xi_n^-)\neq 0$ and, therefore there exists the only solution:

$$c_j^+=-\frac{(-1)^{j-1}V_j(\Theta)\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}(\theta_q-\alpha)}{(p(\alpha)-\lambda)V(\Theta)},\ c_j^-=\frac{(-1)^{n+j-1}V_{n+j}(\Theta)\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}(\theta_q-\alpha)}{(p(\alpha)-\lambda)V(\Theta)},$$

$j=1,...,n$.

Lemma 2.2. *Let $\lambda\in\Lambda_+$, $f_\alpha(x)=e^{i\alpha x}\chi_+(x)$, with $\alpha\in\mathbb{C}_+$ and $\alpha\neq\theta_j(\lambda)$, for $j=1,...,2n$. Then the Fourier transform of $y_\alpha(x,\lambda):=R_A(\lambda)f_\alpha$ is*

$$\widehat{y_\alpha}(t,\lambda)=\frac{1}{i\sqrt{2\pi}}\frac{\prod\limits_{j=1}^{2n}\left(\theta_j(\lambda)-\alpha\right)}{\left(p(\alpha)-\lambda\right)(t-\alpha)\prod\limits_{j=1}^{2n}\left(\theta_j(\lambda)-t\right)}.$$

Proof. Using the expressions for $c_j^\pm$, we have

$$\widehat{y_\alpha}(t)=-\frac{1}{p(\alpha)-\lambda}\,\frac{1}{i\sqrt{2\pi}(\alpha-t)}-\sum_{j=1}^{n}c_j^+\frac{1}{i\sqrt{2\pi}(\xi_j^+-t)}+\sum_{j=1}^{n}c_j^-\frac{1}{i\sqrt{2\pi}(\xi_j^--t)}=$$

$$=\frac{1}{i\sqrt{2\pi}\left(p(\alpha)-\lambda\right)}\left(\frac{1}{(t-\alpha)}+\sum_{j=1}^{2n}\frac{(-1)^{j-1}V_j(\Theta)\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}(\theta_q-\alpha)}{V(\Theta)}\frac{1}{\theta_j-t}\right)=$$

$$=\frac{V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j-t)+\sum\limits_{j=1}^{2n}(-1)^{j-1}V_j(\Theta)(t-\alpha)\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}\left[(\theta_q-\alpha)(\theta_q-t)\right]}{i\sqrt{2\pi}\left(p(\alpha)-\lambda\right)(t-\alpha)V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j-t)}.$$

The numerator of the last fraction is a polynomial $p_1(t)$ of degree $2n$. It is clear that $p_1(\alpha) = p_1(\theta_1) = p_1(\theta_2) = \cdots = p_1(\theta_{2n}) = V(\Theta)\prod_{j=1}^{2n}(\theta_j - \alpha)$. So

$$p_1(t) \equiv V(\Theta)\prod_{j=1}^{2n}(\theta_j - \alpha). \tag{2.4}$$

Finally, we have

$$\widehat{y}(t) = \frac{\prod_{j=1}^{2n}(\theta_j - \alpha)}{i\sqrt{2\pi}(p(\alpha) - \lambda)(t - \alpha)\prod_{j=1}^{2n}(\theta_j - t)}.$$

□

Let us find the resolvents $(A - \lambda I)^{-1}$ and $(A^* - \lambda I)^{-1}$. Since $p(D) = \mathcal{F}^{-1}p(t)\mathcal{F}$, we have

$$y_1^+(x) := (p(D) - \lambda I)^{-1} f_+(x) = \mathcal{F}^{-1}\frac{1}{p(x) - \lambda}\mathcal{F}f_+(x),$$
$$y_1^-(x) := (-p(D) - \lambda I)^{-1} f_-(x) = \mathcal{F}^{-1}\frac{1}{-p(x) - \lambda}\mathcal{F}f_-(x).$$

Since $y_1^+(x)\chi_+(x) \in D(A_0^*)$, $y_1^-(x)\chi_-(x) \in D(A_0^*)$, then $(A_0^* - \lambda I)(y_1^+\chi_+ + y_1^-\chi_-) = f$. Therefore

$$y_f(x) := (A - \lambda I)^{-1}f(x) =$$
$$= y_1^+(x)\chi_+(x) + y_1^-(x)\chi_-(x) + \sum_{j=1}^n c_j^+ e^{i\xi_j^+ x}\chi_+(x) + \sum_{j=1}^n c_j^- e^{i\xi_j^- x}\chi_-(x),$$
$$y_f^*(x) := (A^* - \lambda I)^{-1}f(x) =$$
$$= y_1^+(x)\chi_+(x) + y_1^-(x)\chi_-(x) + \sum_{j=1}^n c_j^{*+} e^{i\xi_j^+ x}\chi_+(x) + \sum_{j=1}^n c_j^{*-} e^{i\xi_j^- x}\chi_-(x).$$

Using (2.2) we have

$$D^q\left(y_1^+(x) + \sum_{j=1}^n c_j^+ e^{i\xi_j^+ x}\right)_{x=0} = D^q\left(y_1^-(x) + \sum_{j=1}^n c_j^- e^{i\xi_j^- x}\right)_{x=0},$$

$q = 0, ..., 2n-1$. Thus, we obtain the system

$$(D^q y_1^+)(0) + \sum_{j=1}^n c_j^+(\xi_j^+)^q = (D^q y_1^-)(0) + \sum_{j=1}^n c_j^-(\xi_j^-)^q,\ q = 0, ..., 2n-1. \tag{2.5}$$

From

$$D^q y_1^+(x) = D^q\mathcal{F}^{-1}\frac{1}{p(x) - \lambda}\mathcal{F}f_+(x) = \mathcal{F}^{-1}x^q\frac{1}{p(x) - \lambda}\mathcal{F}f_+(x),$$

one derives

$$(D^q y_1^+)(0) = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ixt}\, t^q\frac{1}{p(t) - \lambda}\widehat{f_+}(t)\,dt\right)_{x=0} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{t^q\widehat{f_+}(t)}{p(t) - \lambda}\,dt.$$

Analogously, $(D^q y_1^-)(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{t^q \widehat{f_-}(t)}{-p(t)-\lambda}\, dt$. Substituting these equalities into (2.5), we obtain the system

$$M(\Theta(\lambda))\,\mathrm{col}(c_1^+, ..., c_n^+, -c_1^-, ..., -c_n^-) =$$
$$= \mathrm{col}\left(\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \tau^q \left(-\frac{\widehat{f_+}(\tau)}{p(\tau)-\lambda} + \frac{\widehat{f_-}(\tau)}{-p(\tau)-\lambda} \right) d\tau \right)_{q=0}^{2n-1}.$$

The solution is given by

$$c_j^+ = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\theta_q-\tau)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\theta_q-\theta_j)} \left(-\frac{\widehat{f_+}(\tau)}{p(\tau)-\lambda} + \frac{\widehat{f_-}(\tau)}{-p(\tau)-\lambda} \right) d\tau,\ j = 1, ..., n,$$

$$c_{j-n}^- = \frac{-1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\theta_q-\tau)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\theta_q-\theta_j)} \left(-\frac{\widehat{f_+}(\tau)}{p(\tau)-\lambda} + \frac{\widehat{f_-}(\tau)}{-p(\tau)-\lambda} \right) d\tau,\ j = n+1, ..., 2n.$$

Lemma 2.3. *Let $\lambda \in \mathbb{C}_+ \setminus \Lambda_0$ and $f \in L^2(\mathbb{R})$. Then the resolvents $y_f(x,\lambda) = R_A(\lambda)f(x)$ and $y_f^*(x,\lambda) = R_{A^*}(\lambda)f(x)$ are given by the following formulas:*

$$y_f = y_1^+\chi_+ + y_1^-\chi_- + y_0^+ + y_0^-, \tag{2.6}$$
$$y_f^* = y_1^+\chi_+ + y_1^-\chi_- + y_0^+\chi_+ - y_0^+\chi_- - y_0^-\chi_- + y_0^-\chi_-, \tag{2.7}$$

where the functions $y_1^\pm$, $y_0^\pm$ are defined by their Fourier transforms:

$$\widehat{y_1^+}(t) = \frac{\widehat{f_+}(t)}{p(t)-\lambda}\ , \quad \widehat{y_0^+}(t) = -\sum_{j=n+1}^{2n}\sum_{q=1}^{2n} \frac{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}(\theta_r-\xi_j^+)}{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}(\theta_r-\theta_q)} \frac{1}{\theta_q - t}\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)}, \tag{2.8}$$

$$\widehat{y_1^-}(t) = \frac{\widehat{f_-}(t)}{-p(t)-\lambda}\ , \quad \widehat{y_0^-}(t) = \sum_{j=n+1}^{2n}\sum_{q=1}^{2n} \frac{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}(\theta_r-\xi_j^-)}{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}(\theta_r-\theta_q)} \frac{1}{\theta_q - t}\frac{\widehat{f_-}(\xi_j^-)}{p'(\xi_j^-)}. \tag{2.9}$$

Proof. We have shown that $y_f = y_1^+\chi_+ + y_1^-\chi_- + y_0$, where

$$y_0(x) := \sum_{j=1}^n c_j^+ e^{i\xi_j^+ x}\chi_+(x) + \sum_{j=1}^n c_j^- e^{i\xi_j^- x}\chi_-(x).$$

Substituting $c_j^{\pm}$, we have

$$\widehat{y_0}(t) = \sum_{j=1}^{n} c_j^{+} \frac{-1}{i\sqrt{2\pi}(\xi_j^{+} - t)} + \sum_{j=1}^{n} c_j^{-} \frac{1}{i\sqrt{2\pi}(\xi_j^{-} - t)} =$$

$$= \sum_{j=1}^{2n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}(\theta_q - \tau)}{\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}(\theta_q - \theta_j)} \left(\frac{\widehat{f_+}(\tau)}{p(\tau) - \lambda} - \frac{\widehat{f_-}(\tau)}{-p(\tau) - \lambda} \right) d\tau \, \frac{1}{i\sqrt{2\pi}(\theta_j - t)} =$$

$$= \int_{-\infty}^{+\infty} \left(\frac{\sum\limits_{j=1}^{2n}(-1)^{j-1}V_j(\Theta)(t-\tau)\prod\limits_{\substack{q=1\\ q\neq j}}^{2n}[(\theta_q - \tau)(\theta_q - t)] + V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)}{2\pi i V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)(t-\tau)} - \right.$$

$$\left. - \frac{V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)}{2\pi i V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)(t-\tau)} \right) \left(\frac{\widehat{f_+}(\tau)}{p(\tau) - \lambda} - \frac{\widehat{f_-}(\tau)}{-p(\tau) - \lambda} \right) d\tau =$$

$$= \int_{-\infty}^{+\infty} \frac{V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - \tau) - V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)}{2\pi i V(\Theta)\prod\limits_{j=1}^{2n}(\theta_j - t)(t-\tau)} \left(\frac{\widehat{f_+}(\tau)}{p(\tau) - \lambda} - \frac{\widehat{f_-}(\tau)}{-p(\tau) - \lambda} \right) d\tau =$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\prod\limits_{j=1}^{2n}(\theta_j - \tau) - \prod\limits_{j=1}^{2n}(\theta_j - t)}{(t-\tau)\prod\limits_{j=1}^{2n}(\theta_j - t)} \left(\frac{\widehat{f_+}(\tau)}{p(\tau) - \lambda} - \frac{\widehat{f_-}(\tau)}{-p(\tau) - \lambda} \right) d\tau.$$

Here we used the relation (compare (2.4))

$$\sum_{j=1}^{2n}(-1)^{j-1}V_j(\Theta)(t-\tau)\prod_{\substack{q=1\\ q\neq j}}^{2n}[(\theta_q - \tau)(\theta_q - t)] = V(\Theta)\prod_{j=1}^{2n}(\theta_j - \tau) - V(\Theta)\prod_{j=1}^{2n}(\theta_j - t). \quad (2.10)$$

Let us denote

$$\widehat{y_0^{\pm}}(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\prod\limits_{j=1}^{2n}(\theta_j - \tau) - \prod\limits_{j=1}^{2n}(\theta_j - t)}{(t-\tau)\prod\limits_{j=1}^{2n}(\theta_j - t)} \frac{\widehat{f_{\pm}}(\tau)}{p(\tau) \mp \lambda} d\tau.$$

It is easy to see that

$$\widehat{y_0^+}(t) = \frac{1}{2\pi i}\frac{1}{\prod\limits_{j=1}^{2n}(\theta_j - t)}\text{v.p.-}\int\limits_{-\infty}^{+\infty}\frac{\prod\limits_{j=1}^{2n}(\theta_j-\tau)\widehat{f_+}(\tau)}{(t-\tau)(p(\tau)-\lambda)}d\tau - \frac{1}{2\pi i}\text{v.p.-}\int\limits_{-\infty}^{+\infty}\frac{\widehat{f_+}(\tau)}{(t-\tau)(p(\tau)-\lambda)}d\tau =$$

$$= \frac{1}{2i}\frac{1}{\prod\limits_{j=1}^{2n}(\theta_j - t)}\mathcal{H}\,\frac{\prod\limits_{j=1}^{2n}(\theta_j-t)\widehat{f_+}(t)}{p(t)-\lambda} - \frac{1}{2i}\mathcal{H}\,\frac{\widehat{f_+}(t)}{p(t)-\lambda}.$$

Since $\widehat{f_+}(t) \in H^2(\mathbb{C}_-)$, we have

$$\frac{\widehat{f_+}(t)}{\prod\limits_{j=1}^{2n}(t-\xi_j^+)} - \sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)} \in H^2(\mathbb{C}_-),$$

$$\sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)} \in H^2(\mathbb{C}_+).$$

Therefore,

$$\mathcal{H}\,\frac{\widehat{f_+}(t)}{p(t)-\lambda} = \mathcal{H}\,\frac{\widehat{f_+}(t)}{\prod\limits_{j=1}^{2n}(t-\xi_j^+)} =$$

$$= \mathcal{H}\left(\frac{\widehat{f_+}(t)}{\prod\limits_{j=1}^{2n}(t-\xi_j^+)} - \sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)}\right) + \mathcal{H}\sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)} =$$

$$= i\left(\frac{\widehat{f_+}(t)}{\prod\limits_{j=1}^{2n}(t-\xi_j^+)} - \sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)}\right) + (-i)\sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{\prod\limits_{\substack{q=1\\q\neq j}}^{2n}(\xi_j^+-\xi_q^+)(t-\xi_j^+)} =$$

$$= i\,\frac{\widehat{f_+}(t)}{p(t)-\lambda} - 2i\sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)}.$$

Analogously,

$$\mathcal{H}\,\frac{\prod_{q=1}^{2n}(\theta_q - t)\widehat{f_+}(t)}{p(t)-\lambda} =$$

$$= \mathcal{H}\left(\frac{\prod_{q=1}^{2n}(\theta_q - t)\widehat{f_+}(t)}{p(t)-\lambda} - \sum_{j=n+1}^{2n}\frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)}\right) + \mathcal{H}\sum_{j=n+1}^{2n}\frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)} =$$

$$= i\frac{\prod_{q=1}^{2n}(\theta_q - t)\widehat{f_+}(t)}{p(t)-\lambda} - 2i\sum_{j=n+1}^{2n}\frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)}.$$

Thus,

$$\widehat{y_0^+}(t) = -\frac{1}{2i}\left(i\,\frac{\widehat{f_+}(t)}{p(t)-\lambda} - 2i\sum_{j=n+1}^{2n}\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)}\right) +$$

$$+\frac{1}{2i}\frac{1}{\prod_{j=1}^{2n}(\theta_j - t)}\left(i\,\frac{\prod_{q=1}^{2n}(\theta_q - t)\widehat{f_+}(t)}{p(t)-\lambda} - 2i\sum_{j=n+1}^{2n}\frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(t-\xi_j^+)}\right) =$$

$$= -\sum_{j=n+1}^{2n}\frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^+) - \prod_{j=1}^{2n}(\theta_j - t)}{\prod_{j=1}^{2n}(\theta_j - t)\,(t-\xi_j^+)}\,\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)}.$$

Taking (2.10) into account, we have

$$\widehat{y_0^+}(t) = -\sum_{j=n+1}^{2n}\frac{\sum_{q=1}^{2n}(-1)^{q-1}V_q(\Theta)(t-\xi_j^+)\prod_{\substack{r=1\\ r\neq q}}^{2n}\left[\left(\theta_r - \xi_j^+\right)(\theta_r - t)\right]}{V(\Theta)\prod_{q=1}^{2n}(\theta_q - t)\,(t-\xi_j^+)}\,\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)} =$$

$$= -\sum_{j=n+1}^{2n}\sum_{q=1}^{2n}\frac{\prod_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \xi_j^+)}{\prod_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \theta_q)(\theta_q - t)}\,\frac{\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)}\ .$$

In just the same way one derives

$$\widehat{y_0^-}(t) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\prod_{j=1}^{2n}(\theta_j-\tau) - \prod_{j=1}^{2n}(\theta_j - t)}{(t-\tau)\prod_{j=1}^{2n}(\theta_j - t)} \frac{\widehat{f_-}(\tau)}{p(\tau)+\lambda}\, d\tau =$$

$$= \sum_{j=n+1}^{2n} \frac{\prod_{q=1}^{2n}(\theta_q - \xi_j^-) - \prod_{j=1}^{2n}(\theta_j - t)}{\prod_{j=1}^{2n}(\theta_j - t)\,(t-\xi_j^-)} \frac{\widehat{f_-}(\xi_j^-)}{p'(\xi_j^-)} =$$

$$= \sum_{j=n+1}^{2n}\sum_{q=1}^{2n} \frac{\prod_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \xi_j^-)}{\prod_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \theta_q)(\theta_q - t)} \frac{\widehat{f_-}(\xi_j^-)}{p'(\xi_j^-)}.$$

Since $\widehat{y_0} = \widehat{y_0^+} + \widehat{y_0^-}$, we conclude (2.6).

We can obtain (2.7) similarly. On the other hand, note that the functions $y_0^+\chi_+$, $y_0^+\chi_-$, $y_0^-\chi_+$, $y_0^-\chi_-$ belong to $\ker(A_0^* - \lambda I)$ and

$$y^* := y_1^+\chi_+ + y_1^-\chi_- + y_0^+\chi_+ - y_0^+\chi_- - y_0^-\chi_- + y_0^-\chi_- \in D(A^*).$$

It follows that

$$(A^* - \lambda I)y^* = (A_0^* - \lambda I)y^* = (A_0^* - \lambda I)(y_1^+\chi_+ + y_1^-\chi_-) = f_+ + f_- = f\ ,$$

and, therefore $y_f^* = y^*$. □

3 Necessary conditions for similarity

Consider the case when the polynomial $p(t)$ changes sign. Then the polynomial has real roots of odd multiplicity. If $\lambda = 0$, then the equation $p(\xi) - \lambda = 0$ coincides with the equation $p(\xi) + \lambda = 0$. For this reason we investigate the behavior of the resolvent $(A - \lambda I)^{-1}$ for λ small enough.

Recal that $\Xi^\pm = \{\xi_j^\pm(\lambda)\}_1^{2n}$ are the complete solutions (see Section 1) of the equations $p(\xi) \mp \lambda = 0$ enumerated as in the statement of Lemma 1.2, $\Xi := \Xi^+ \cup \Xi^-$, $\Xi_1^+ = \{\xi_j^+\}_1^n$, $\Xi_2^+ = \{\xi_j^+\}_{n+1}^{2n}$, $\Xi_1^- = \{\xi_j^-\}_1^n$, $\Xi_2^- = \{\xi_j^-\}_{n+1}^{2n}$. For an arbitrary family $\Phi \subset \Xi$, $\Phi[z_0, \lambda_0]$ denotes the family of all functions $\xi(\lambda)$ defined on $\overline{\Lambda_+}$ such that $\xi \in \Phi$ and $\xi(\lambda_0) = z_0$. The notation $\sharp(\Phi)$ means the number of elements in a family Φ.

Proposition 3.1. *Assume that $p(t)$ takes both positive and negative values for $t \in \mathbb{R}$. Then there exists $t_0 \in \mathbb{R}$ such that*

$$\sharp(\Xi_1^+[t_0, 0]) = \sharp(\Xi_1^-[t_0, 0]) = k_1 + 1,\ \sharp(\Xi_2^+[t_0, 0]) = \sharp(\Xi_2^-[t_0, 0]) = k_1.$$

where $k_1 = \frac{1}{2}(k(t_0, 0) - 1)$ and $k(t_0, 0)$ is an odd number.

Proof. By definition, $\sum\limits_{z\in\mathbb{C}} \sharp(\Xi_1^{\pm}[z,0]) = n = \sum\limits_{z\in\mathbb{C}} \sharp(\Xi_2^{\pm}[z,0])$. Since $p(z)$ has real coefficients, $\sharp(\Xi_1^{\pm}[z,0]) = \sharp(\Xi_2^{\pm}[\overline{z},0])$ for $z \notin \mathbb{R}$. According to (1.3), we have

$$\sum_{z\notin\mathbb{R}} \sharp(\Xi_1^{+}[z,0]) = \sum_{z\notin\mathbb{R}} \sharp(\Xi_2^{+}[z,0]) = \sum_{z\notin\mathbb{R}} \sharp(\Xi_2^{-}[z,0]) = \sum_{z\notin\mathbb{R}} \sharp(\Xi_1^{-}[z,0]).$$

This implies

$$\sum_{z\in\mathbb{R}} \sharp(\Xi_1^{+}[z,0]) = \sum_{z\in\mathbb{R}} \sharp(\Xi_2^{+}[z,0]) = \sum_{z\in\mathbb{R}} \sharp(\Xi_2^{-}[z,0]) = \sum_{z\in\mathbb{R}} \sharp(\Xi_1^{-}[z,0]). \tag{3.1}$$

Because $p(t)$ changes sign, there exist real roots of odd multiplicity. By Lemma 1.5 one of the assertions (i) or (ii) is valid for such roots. By virtue of (3.1), the both cases are realized. □

Theorem 3.1. *If the polynomial $p(t)$ takes both positive and negative values for $t \in \mathbb{R}$, then the operator $A = (\operatorname{sgn} x)p(D)$ is not similar to selfadjoint operator.*

Proof. Assume that $p(t)$ takes both positive and negative values for $t \in \mathbb{R}$. By Proposition 3.1 there exists $t_0 \in \mathbb{R}$ such that

$$\sharp(\Xi_1^{+}[t_0,0]) = \sharp(\Xi_1^{-}[t_0,0]) = k_1 + 1,\ \sharp(\Xi_2^{+}[t_0,0]) = \sharp(\Xi_2^{-}[t_0,0]) = k_1,$$

where $k_1 = \frac{1}{2}\left(k(t_0,0) - 1\right)$. In this proof we abbreviate $k(t_0,0)$ as $k(t_0)$ and $\sharp(\Theta[t_0,0])$ as $n_\Theta(t_0)$. According to the definition of Θ, we have

$$n_\Theta(t_0) = \sharp(\Xi_1^{+}[t_0,0]) + \sharp(\Xi_1^{-}[t_0,0]) = 2k_1 + 2 \geq 2k_1 + 1 = k(t_0). \tag{3.2}$$

Let $\lambda = \mu + i\varepsilon$, $\operatorname{Im}\lambda = \varepsilon$. In what follows we fix some $\varepsilon \in (0,\varepsilon_1)$ and take integrals along the line $\lambda = \mu + i\varepsilon$, $\mu \in \mathbb{R}$.

Lemma 2.2 and Parseval equality yield the following relation:

$$\varepsilon \int_{-\infty}^{+\infty} \|y_\alpha(x,\lambda)\|_{L^2}^2 d\mu = \varepsilon \int_{-\infty}^{+\infty} \|\widehat{y_\alpha}(t,\lambda)\|_{L^2}^2 d\mu =$$

$$= \frac{\varepsilon}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left| \frac{\prod\limits_{j=1}^{2n} (\theta_j(\lambda) - \alpha)}{(p(\alpha) - \lambda)(t - \alpha) \prod\limits_{j=1}^{2n} (\theta_j(\lambda) - t)} \right|^2 dt\, d\mu \geq$$

$$\geq \frac{\varepsilon}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left| \frac{\prod\limits_{j=1}^{2n} (\theta_j(\lambda) - \alpha)}{(p(\alpha) - \lambda)(t - \alpha)} \right|^2 \left| \frac{1}{\prod\limits_{j=1}^{2n} (|\theta_j(\lambda) - t_0| + |t - t_0|)} \right|^2 dt\, d\mu.$$

Let $T := [t_0, t_0 + \delta_t)$ be a neighborhood of t_0 which does not contain other roots of the equation $p(t)$. By Lemmas 1.1 and 1.4 there exists a neighborhood of zero $B_0 := \{\lambda : 0 < \operatorname{Im}\lambda < \delta_\varepsilon,\ 0 < \operatorname{Re}\lambda < \delta_\mu\}$ such that

$$\begin{aligned} |\theta(\lambda) - t_0| &< C_1 |\lambda|^{\frac{1}{k(t_0)}} \quad \text{for all } \theta \in \Theta[t_0,0], \\ |\theta(\lambda) - t_0| &< C_2 \qquad \text{for all } \theta \in \Theta \setminus \Theta[t_0,0], \end{aligned} \tag{3.3}$$

here C_1, C_2 are some constants.

We choose $\alpha \in \mathbb{C}_+$ such that

$$\left| \frac{\prod_{j=1}^{2n} (\theta_j(\lambda) - \alpha)}{(p(\alpha) - \lambda)\,(t - \alpha)} \right| \geq C_3 = \text{const} > 0 \tag{3.4}$$

for all $\lambda \in B_0$, $t \in T$, here C_3 is some constant.

By virtue of (3.4) we have

$$\varepsilon \int_{-\infty}^{+\infty} \|y_\alpha(x, \lambda)\|_{L^2}^2 d\mu \geq$$

$$\geq \frac{\varepsilon}{2\pi} \int_0^{\delta_\mu} \int_{t_0}^{t_0+\delta_t} \left| \frac{\prod_{j=1}^{2n} (\theta_j(\lambda) - \alpha)}{(p(\alpha) - \lambda)\,(t - \alpha)} \right|^2 \left| \frac{1}{\prod_{j=1}^{2n} (|\theta_j(\lambda) - t_0| + |t - t_0|)} \right|^2 dt\, d\mu \geq$$

$$\geq C_4 \varepsilon \int_0^{\delta_\mu} \int_{t_0}^{t_0+\delta_t} \left| \frac{1}{\prod_{\theta_j \notin \Theta[t_0,0]} (|\theta_j(\lambda) - t_0| + |t - t_0|) \prod_{\theta_j \in \Theta[t_0,0]} (|\theta_j(\lambda) - t_0| + |t - t_0|)} \right|^2 dt\, d\mu,$$

with some constant $C_4 > 0$. Accounting (3.3), for some positive constant C_5, one derives

$$\varepsilon \int_{-\infty}^{+\infty} \|y_\alpha(x, \lambda)\|_{L^2}^2 d\mu \geq$$

$$\geq C_4 \varepsilon \int_0^{\delta_\mu} \int_{t_0}^{t_0+\delta_t} \frac{1}{(C_2 + |t - t_0|)^{2(2n - n_\Theta(t_0))} \left(C_1 |\lambda|^{\frac{1}{k(t_0)}} + |t - t_0| \right)^{2n_\Theta(t_0)}} dt\, d\mu \geq$$

$$\geq C_5 \varepsilon \int_0^{\delta_\mu} \int_{t_0}^{t_0+\delta_t} \frac{1}{\left((t - t_0) + C_1 |\lambda|^{\frac{1}{k(t_0)}} \right)^{2n_\Theta(t_0)}} dt\, d\mu =$$

$$= \frac{C_5 \varepsilon}{2n_\Theta(t_0) - 1} \int_0^{\delta_\mu} \left(\frac{1}{(C_1 |\lambda|)^{\frac{2n_\Theta(t_0) - 1}{k(t_0)}}} - \frac{1}{\left(\delta_t + C_1 |\lambda|^{\frac{1}{k(t_0)}} \right)^{2n_\Theta(t_0) - 1}} \right) d\mu. \tag{3.5}$$

It is clear that

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{\delta_\mu} \frac{1}{\left(\delta_t + C_1 |\lambda|^{\frac{1}{k(t_0)}} \right)^{2n_\Theta(t_0) - 1}} d\mu = 0. \tag{3.6}$$

Further, one has

$$\varepsilon\int\limits_0^{\delta_\mu}\frac{1}{|\lambda|^{\frac{2n_\Theta(t_0)-1}{k(t_0)}}}d\mu=\int\limits_0^{\delta_\mu}\frac{\varepsilon}{(\mu^2+\varepsilon^2)^{\frac{n_\Theta(t_0)-1/2}{k(t_0)}}}d\mu=$$

$$=\varepsilon^{2\left(1-\frac{n_\Theta(t_0)-1/2}{k(t_0)}\right)}\int\limits_0^{\delta_\mu}\frac{1}{\left(\left(\frac{\mu}{\varepsilon}\right)^2+1\right)^{\frac{n_\Theta(t_0)-1/2}{k(t_0)}}}d\left(\frac{\mu}{\varepsilon}\right)=$$

$$=\varepsilon^{2\left(1-\frac{n_\Theta(t_0)-1/2}{k(t_0)}\right)}\int\limits_0^{\delta_\mu/\varepsilon}\frac{1}{(\mu_1^2+1)^{\frac{n_\Theta(t_0)-1/2}{k(t_0)}}}d\mu_1. \tag{3.7}$$

Besides,

$$\lim_{\varepsilon\to 0}\int\limits_0^{\delta_\mu/\varepsilon}\frac{1}{(\mu_1^2+1)^{\frac{n_\Theta(t_0)-1/2}{k(t_0)}}}d\mu_1=\int\limits_0^{+\infty}\frac{1}{(\mu_1^2+1)^{\frac{n_\Theta(t_0)-1/2}{k(t_0)}}}d\mu_1>0. \tag{3.8}$$

By virtue of (3.2) it easy to see that $\dfrac{n_\Theta(t_0)-1/2}{k(t_0)}=\dfrac{2k_1+3/2}{2k_1+1}>1$ and, consequently, $\varepsilon^{2\left(1-\frac{n_\Theta(t_0)-1/2}{k(t_0)}\right)}\to+\infty$ as $\varepsilon\to 0$. Therefore, it follows from (3.7), (3.8) that

$$\lim_{\varepsilon\to 0}\varepsilon\int\limits_0^{\delta_\mu}\frac{1}{|\lambda|^{\frac{2n_\Theta(t_0)-1}{k(t_0)}}}d\mu=+\infty. \tag{3.9}$$

Now, the relation $\lim\limits_{\varepsilon\to 0}\varepsilon\int\limits_{-\infty}^{+\infty}\|R_A(\lambda)f_\alpha\|^2_{L^2}d\mu=+\infty$ is implied by (3.5), (3.6), (3.9). Thus, it is shown that inequality (0.2) of Theorem 0.2 is not valid for $f_\alpha(x)=e^{i\alpha x}\chi_+(x)\in L^2(\mathbb{R})$. Hence the operator A is not similar to a selfadjoint operator. □

Remark 3.1. Let $p(z)$ be an odd order polynomial. It is easy to see that the spectrum of A is the whole complex plane, $\sigma(A)=\mathbb{C}$. Therefore, the operator $A=Jp(D)$ is not similar to selfadjoint one.

4 Sufficient conditions for similarity

In Section 2 we have shown, that the operator A is closed and the spectrum of A is real. To prove that A is similar to a selfadjoint operator we have to check expressions (0.2), (0.3). The formulas for the resolvents of A and A^* are provided in Lemma 2.3.

There follows from (2.8) and from Parseval equality that

$$\varepsilon \int_{-\infty}^{+\infty} \|y_1^+(x,\lambda)\chi_+(x)\|_{L^2}^2 d\mu \le \varepsilon \int_{-\infty}^{+\infty} \|\widehat{y_1^+}(t,\lambda)\|_{L^2}^2 d\mu =$$

$$= \varepsilon \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\frac{\widehat{f_+}(t)}{p(t)-\lambda}\right|^2 dt\, d\mu = \varepsilon \int_{-\infty}^{+\infty} |\widehat{f_+}(t)|^2 \int_{-\infty}^{+\infty} \frac{1}{(p(t)-\mu)^2+\varepsilon^2} d\mu\, dt =$$

$$= \pi\|\widehat{f_+}(t)\|_{L^2}^2 = \pi\|f_+(x)\|_{L^2}^2. \tag{4.1}$$

In the same way one obtains

$$\varepsilon \int_{-\infty}^{+\infty} \|y_1^-(x,\lambda)\|_{L^2}^2 d\mu \le \pi\|f_-(x)\|_{L^2}^2. \tag{4.2}$$

Remark 4.1. On the other hand

$$y_1^+(x,\lambda) = R_{p(D)}(\lambda)\, f(x),\;\; y_1^-(x,\lambda) = R_{-p(D)}(\lambda)\, f(x).$$

Since $p(D)$ is a selfadjoint operator, estimates (4.1), (4.2) are consequences of Theorem 0.2.

For $n+1 \le j \le 2n$, $1 \le q \le 2n$ we define functions $y^{\pm}(x,\lambda)$ by their Fourier transforms:

$$\widehat{y_{jq}^{\pm}}(t,\lambda) := \mp \frac{\prod\limits_{\substack{r=1\\ r\ne q}}^{2n}(\theta_r-\xi_j^{\pm})}{\prod\limits_{\substack{r=1\\ r\ne q}}^{2n}(\theta_r-\theta_q)} \frac{\widehat{f_+}(\xi_j^{\pm})}{p'(\xi_j^{\pm})} \frac{1}{\theta_q - t}.$$

Then $\widehat{y_0^{\pm}} = \sum\limits_{j=n+1}^{2n}\sum\limits_{q=1}^{2n} \widehat{y_{jq}^{\pm}}$.

Proposition 4.1. *Let $\mathcal{M}$ be a measurable subset of the real line, $\lambda = \mu + i\varepsilon$. Then*

$$\varepsilon \int_{\mathcal{M}} \|y_{jq}^{\pm}(x,\lambda)\|_{L^2}^2 d\mu \le \pi M_C \sup_{\mu\in\mathcal{M}} B_{jq}^{\pm}(\mu+i\varepsilon)\, \|f_{\pm}(x)\|_{L^2}^2, \tag{4.3}$$

where

$$B_{jq}^{\pm}(\lambda) := \frac{\operatorname{Im}\lambda \prod\limits_{\substack{r=1\\ r\ne q}}^{2n} |\theta_r(\lambda)-\xi_j^{\pm}(\lambda)|^2}{\prod\limits_{\substack{r=1\\ r\ne q}}^{2n} |\theta_r(\lambda)-\theta_q(\lambda)|^2 \left|p'\left(\xi_j^{\pm}(\lambda)\right)\right|} \frac{1}{|\operatorname{Im}\theta_q(\lambda)|}.$$

(The constant M_C was defined in Lemma 1.6.)

Proof. We prove the proposition for y_{jq}^+. The other case is analogous.

By definition

$$\varepsilon \int\limits_{\mathcal{M}} \|\widehat{y_{jq}^{+}}(t,\lambda)\|_{L^2}^2 d\mu = \varepsilon \int\limits_{\mathcal{M}} \frac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n} |\theta_r(\lambda)-\xi_j^+(\lambda)|^2}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n} |\theta_r(\lambda)-\theta_q(\lambda)|^2 \left|p'\left(\xi_j^+(\lambda)\right)\right|^2} \left\|\frac{1}{\theta_q(\lambda)-t}\right\|_{L^2}^2 \left|\widehat{f_+}\left(\xi_j^+(\lambda)\right)\right|^2 d\mu.$$

Since $\left(\xi_j^+(\lambda)\right)' = \dfrac{1}{p'\left(\xi_j^+(\lambda)\right)}$, we have

$$\varepsilon \int\limits_{\mathcal{M}} \|\widehat{y_{jq}^{+}}(t,\lambda)\|_{L^2}^2 d\mu =$$

$$= \pi \int\limits_{\mathcal{M}} \frac{\operatorname{Im}\lambda \prod\limits_{\substack{r=1\\ r\neq q}}^{2n} |\theta_r(\lambda)-\xi_j^+(\lambda)|^2}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n} |\theta_r(\lambda)-\theta_q(\lambda)|^2 \left|p'\left(\xi_j^+(\lambda)\right)\right|} \frac{1}{|\operatorname{Im}\theta_q(\lambda)|} \left|\widehat{f_+}\left(\xi_j^+(\lambda)\right)\right|^2 \left|\tfrac{d\xi_j^+(\mu+i\varepsilon)}{d\mu}\right| d\mu =$$

$$\leq \pi \sup_{\mu\in\mathcal{M}} B_{jq}^+(\mu+i\varepsilon) \int\limits_{\mu\in\mathbb{R}} \left|\widehat{f_+}\left(\xi_j^+(\lambda)\right)\right|^2 \left|d\xi_j^+(\mu+i\varepsilon)\right|.$$

By Lemma 1.6 we arrive at inequality (4.3). □

To estimate the functions $B_{jq}^{\pm}(\lambda)$ we derive several propositions from lemmas of Section 1.

Let $\omega_1(\lambda)$, $\omega_2(\lambda)$ be arbitrary different functions from the family $\{\omega_{k,j}(\lambda), \omega_{k,k-j+1}(-\lambda)\}_{j=1}^{k}$. Then $|\omega_1(\lambda)| = |\omega_2(\lambda)| = |\lambda|^{\frac{1}{k}}$. Since the angle $\alpha := \arg\omega_1(\lambda) - \arg\omega_2(\lambda) \neq 0$ is constant and is a multiple of π/k, we have

$$|\omega_1(\lambda)-\omega_2(\lambda)| = 2\sin\frac{\alpha}{2}|\lambda|^{\frac{1}{k}}. \tag{4.4}$$

Proposition 4.2. *(a) Let z_0 be a root of the polynomial $p(\xi)\mp\lambda_0$ of multiplicity $k = k(z_0,\pm\lambda_0) \geq 2$ and $\left\{\tilde{\xi}_q^{\pm}(\lambda)\right\}_1^k = \Xi^{\pm}[z_0,\lambda_0]$. Then, for λ sufficiently close to λ_0 and for some positive constants $M_1(\lambda_0)$, $M_2(\lambda_0)$ the following inequality holds:*

$$M_1(\lambda_0)\,|\lambda-\lambda_0|^{\frac{1}{k}} < \left|\tilde{\xi}_j^{\pm}(\lambda)-\tilde{\xi}_q^{\pm}(\lambda)\right| < M_2(\lambda_0)\,|\lambda-\lambda_0|^{\frac{1}{k}}\ ,\ j\neq q\ . \tag{4.5}$$

(b) If $\lambda_0\neq 0$, $\xi^+(\lambda)\in\Xi^+$ and $\xi^-(\lambda)\in\Xi^-$, then $\xi^+(\lambda_0)\neq\xi^-(\lambda_0)$.

(c) Let z_0 be a root of the polynomial $p(\xi)$ of multiplicity $k = k(z_0,0)$. Then

$$\sharp(\Xi^+[z_0,0]) = \sharp(\Xi^-[z_0,0]) = k, \quad \sharp(\Xi[z_0,0]) = \sharp(\Xi^+[z_0,0]) + \sharp(\Xi^-[z_0,0]) = 2k.$$

For any different functions $\xi_1(\lambda)$, $\xi_2(\lambda)$ from $\Xi[z_0,0]$ and for λ sufficiently close to λ_0 the following inequality holds:

$$M_1(0)\,|\lambda|^{\frac{1}{k}} < |\xi_1(\lambda)-\xi_2(\lambda)| < M_2(0)\,|\lambda|^{\frac{1}{k}}\ , \tag{4.6}$$

where $M_1(0)$, $M_2(0)$ are some positive constants.

(d) Let $\xi_1(\lambda)$, $\xi_2(\lambda)$ be arbitrary different functions from the family Ξ. Then, for λ large enough and for some positive constants $M_1(\infty)$, $M_2(\infty)$ the following inequality holds:

$$M_1(\infty)\,|\lambda|^{\frac{1}{2n}} < |\xi_1(\lambda)-\xi_2(\lambda)| < M_2(\infty)\,|\lambda|^{\frac{1}{2n}}\ . \tag{4.7}$$

Remark 4.2. Assume $\xi_1(\lambda) \in \Xi^+$, $\xi_2(\lambda) \in \Xi^-$. Then the equlity $\xi_1(0) = \xi_2(0)$ is possible. So we need item (c) of Proposition 4.2 to estimate the difference $\xi_1(\lambda) - \xi_2(\lambda)$ for λ small enough.

Proof. The statement (a) follows at once from (4.4) and Lemma 1.4.

(b) Since $p(\xi^+(\lambda_0)) - \lambda_0 = p(\xi^-(\lambda_0)) + \lambda_0 = 0$, $\xi^+(\lambda_0) = \xi^-(\lambda_0)$ implies $\lambda_0 = -\lambda_0 = 0$.

(c) Let $\lambda_0 = 0$. Then we can choose the function $\gamma(z)$ in Lemma 1.4 such that the both equalities

$$\tilde{\xi}_q^+(\lambda) - z_0 = \gamma\left(\tilde{\xi}_q^+(\lambda)\right)\omega_{k,q}(\lambda - \lambda_0),\ \tilde{\xi}_q^-(\lambda) - z_0 = \gamma\left(\tilde{\xi}_q^-(\lambda)\right)\omega_{k,q}(-\lambda + \lambda_0)$$

hold in a neighborhood of λ_0. Accounting (4.4), we obtain (c).

The statement (d) follows at once from (4.4) and Lemma 1.2. □

Proposition 4.3. *Let $\xi(\lambda) \in \Xi$.*

(a) The function $\dfrac{\operatorname{Im}\lambda}{\operatorname{Im}\xi(\lambda)\,|\lambda|^{\frac{2n-1}{2n}}}$ *is bounded in a neighborhood of $\lambda_0 = \infty$.*

(b) If $\lambda_0 \in \mathbb{R}$ and $\xi(\lambda_0) = z_0 \in \mathbb{R}$, then the function $\dfrac{\operatorname{Im}\lambda}{\operatorname{Im}\xi(\lambda)\,|\lambda - \lambda_0|^{\frac{k(z_0,\lambda_0)-1}{k(z_0,\lambda_0)}}}$ *is bounded in a neighborhood of λ_0.*

Proof. Let $\xi(\lambda) \in \Xi^+$. The case $\xi(\lambda) \in \Xi^-$ is considered analogously.

(a) Since $p(\xi(\lambda)) = \sum_{j=0}^{2n} a_j(\xi(\lambda))^j = \lambda$ and $a_j \in \mathbb{R}$, we have

$$\operatorname{Im}\lambda = \operatorname{Im}\sum_{j=0}^{2n} a_j(\xi(\lambda))^j = \sum_{j=1}^{2n} a_j \operatorname{Im}(\operatorname{Re}\xi(\lambda) + i\operatorname{Im}\xi(\lambda))^j\ .$$

Removing the parentheses, one obtains

$$\operatorname{Im}(\operatorname{Re}\xi(\lambda) + i\operatorname{Im}\xi(\lambda))^j = \operatorname{Im}\xi(\lambda)\ p_{j-1}(\operatorname{Re}\xi(\lambda), \operatorname{Im}\xi(\lambda))\ ,$$

where p_{j-1} is a polynomial of degree $j-1$ in two variables. So we have

$$\operatorname{Im}\lambda = \operatorname{Im}\xi(\lambda)\sum_{j=1}^{2n} a_j\ p_{j-1}(\operatorname{Re}\xi(\lambda), \operatorname{Im}\xi(\lambda)) = \operatorname{Im}\xi(\lambda)\ P_{2n-1}(\operatorname{Re}\xi(\lambda), \operatorname{Im}\xi(\lambda))\ ,$$

where P_{2n-1} is a polynomial of degree $2n-1$ in two variables.

According to Lemma 1.2, the inequalities $|\operatorname{Re}\xi(\lambda)| < |\xi(\lambda)| < C|\lambda|^{1/2n}$, $|\operatorname{Im}\xi(\lambda)| < |\xi(\lambda)| < C|\lambda|^{1/2n}$ are valid for λ large enough and for some constant $C > 0$. Consequently, the function $\dfrac{\operatorname{Im}\lambda}{\operatorname{Im}\xi(\lambda)|\lambda|^{\frac{2n-1}{2n}}} = \dfrac{P_{2n-1}(\operatorname{Re}\xi(\lambda), \operatorname{Im}\xi(\lambda))}{|\lambda|^{\frac{2n-1}{2n}}}$ is bounded in a neighbourhood of $\lambda_0 = \infty$.

(b) Since z_0 is a real root of multiplicity $k = k(z_0, \lambda_0)$ of the polynomial $p(\xi) - \lambda_0$, we can write $p(\xi) - \lambda_0 = (\xi - z_0)^k \tilde{p}(\xi)$, where $\tilde{p}(\xi)$ is a polynomial of degree $2n-k$ with real coefficients. Then

$$0 = p(\xi(\lambda)) - \lambda_0 - (\lambda - \lambda_0) = (\xi(\lambda) - z_0)^k\,\tilde{p}(\xi(\lambda)) - (\lambda - \lambda_0).$$

Since $\lambda_0 \in \mathbb{R}$, we have

$$\operatorname{Im}\lambda = \operatorname{Im}(\lambda - \lambda_0) = \operatorname{Im}(\xi(\lambda) - z_0)^k \operatorname{Re}\tilde{p}(\xi(\lambda)) + \operatorname{Re}(\xi(\lambda) - z_0)^k \operatorname{Im}\tilde{p}(\xi(\lambda))\ .$$

Note that

$$\operatorname{Im}(\xi(\lambda)-z_0)^k = \operatorname{Im}\Big(\operatorname{Re}(\xi(\lambda)-z_0)+i\operatorname{Im}\xi(\lambda)\Big)^k =$$
$$= \operatorname{Im}\xi(\lambda)\, P_{k-1}\Big(\operatorname{Re}(\xi(\lambda)-z_0)\,, i\operatorname{Im}\xi(\lambda)\Big)\,, \quad (4.8)$$

where P_{k-1} is a polynomial in two variables with monomials of degree $k-1$. According to Lemma 1.4, the inequalities

$$|\operatorname{Re}(\xi(\lambda)-z_0)| < |\xi(\lambda)-z_0| < C|\lambda-\lambda_0|^{1/k}\,,$$
$$|\operatorname{Im}\xi(\lambda)| = |\operatorname{Im}(\xi(\lambda)-z_0)| < |\xi(\lambda)-z_0| < C|\lambda-\lambda_0|^{1/k}$$

hold for λ in a neighborhood of λ_0 and for some constant $C>0$. Therefore, bearing in mind (4.8), the function

$$b_1(\lambda) := \frac{\operatorname{Im}(\xi(\lambda)-z_0)^k \operatorname{Re}\tilde p(\xi(\lambda))}{\operatorname{Im}\xi(\lambda)|\lambda-\lambda_0|^{\frac{k-1}{k}}} = \frac{P_{k-1}\Big(\operatorname{Re}(\xi(\lambda)-z_0)\,, i\operatorname{Im}\xi(\lambda)\Big)\operatorname{Re}\tilde p(\xi(\lambda))}{|\lambda-\lambda_0|^{\frac{k-1}{k}}}$$

is bounded in a neighborhood of λ_0.

Besides,

$$\operatorname{Re}(\xi(\lambda)-z_0)^k \operatorname{Im}\tilde p(\xi(\lambda)) = \operatorname{Re}(\xi(\lambda)-z_0)^k \operatorname{Im}\xi(\lambda)\,\tilde q\Big(\operatorname{Re}(\xi(\lambda)-z_0)\,, \operatorname{Im}\xi(\lambda)\Big)\,,$$

where $\tilde q$ is a polynomial in two variables. Since the inequalities

$$|\operatorname{Re}(\xi(\lambda)-z_0)^k| < |\xi(\lambda)-z_0|^k < C|\lambda-\lambda_0|^{k/k} = C|\lambda-\lambda_0|$$

are valid for λ in a neighborhood of λ_0, we have $\lim\limits_{\lambda\to\lambda_0} b_2(\lambda)=0$ for the function

$$b_2(\lambda) := \frac{\operatorname{Re}(\xi(\lambda)-z_0)^k \operatorname{Im}\tilde p(\xi(\lambda))}{\operatorname{Im}\xi(\lambda)|\lambda-\lambda_0|^{\frac{k-1}{k}}} = \frac{\operatorname{Re}(\xi(\lambda)-z_0)^k\,\tilde q\Big(\operatorname{Re}(\xi(\lambda)-z_0)\,, \operatorname{Im}\xi(\lambda)\Big)}{|\lambda-\lambda_0|^{\frac{k-1}{k}}}.$$

Hence, the function $b_1(\lambda)+b_2(\lambda) = \dfrac{\operatorname{Im}\lambda}{\operatorname{Im}\xi(\lambda)|\lambda-\lambda_0|^{\frac{k-1}{k}}}$ is bounded in a neighborhood of λ_0. □

For all $\xi_j^{\pm}\in\Xi$ we define the sets

$$\Lambda(\xi_j^+) = \Big\{\lambda_0\in\Lambda_0\cap\overline{(C)_+} : k\left(\xi_j^+(\lambda_0),\lambda_0\right)>1,\ \xi_j^+(\lambda_0)\notin\mathbb{R}\Big\}\,,\ j=1,...,2n,$$
$$\Lambda(\xi_j^-) = \Big\{\lambda_0\in\Lambda_0\cap\overline{(C)_+} : k\left(\xi_j^-(\lambda_0),-\lambda_0\right)>1,\ \xi_j^+(\lambda_0)\notin\mathbb{R}\Big\}\,,\ j=1,...,2n.$$

If $1\le j\le n$ then $\theta_j=\xi_j^+$, therefore $\Lambda(\theta_j)=\Lambda(\xi_j^+)$. If $n+1\le j\le 2n$, then $\theta_j=\xi_{j-n}^-$, therefore $\Lambda(\theta_j)=\Lambda(\xi_{j-n}^-)$.

Lemma 4.1. *(a) Let $\xi(\lambda)\in\Xi_2^+\cup\Xi_2^-$, $\lambda_0\in\overline{\Lambda_+}\setminus\Lambda(\xi)$ and $\lambda_0\neq\infty$. If $\theta_q(\lambda_0)\neq\xi(\lambda_0)$, the function $\dfrac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}|\theta_r(\lambda)-\xi(\lambda)|^2}{p'(\xi(\lambda))}$ is bounded in a neighbourhood of λ_0.*

(b) Let $\lambda_0 = \infty$ or $\lambda_0 \in \overline{\Lambda_+} \setminus \Lambda(\theta_q)$, $\lambda_0 \neq 0$. Then the function $\dfrac{\operatorname{Im}\lambda}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n} |\theta_r(\lambda)-\theta_q(\lambda)|^2 \operatorname{Im}\theta_q(\lambda)}$ *is bounded in a neighbourhood of λ_0.*

(c) Let the polynomial $p(t)$ be nonnegative. Let $\lambda_0 \in \{0,\infty\}$ or $\lambda_0 \in \overline{\Lambda_+} \setminus \big(\Lambda(\theta_q)\cup\Lambda(\xi_j^{\pm})\big)$, $n+1 \le j \le 2n$, $1 \le q \le 2n$. Then the function $B_{jq}^{\pm}(\lambda)$ is bounded in a neighbourhood of λ_0.

Proof. We prove statement (a) for $\xi \in \Xi_2^+$, statement (b) for $\theta_q \in \Xi_1^+$, statement (c) for a function B_{jq}^+. The remaining cases are considered analogously.

If $\lambda_0 \in \overline{\Lambda_+} \setminus \Lambda(\xi_j^+)$, then either $z_0 := \xi_j^+(\lambda_0)$ is a simple root of the equation $p(z)-\lambda_0 = 0$, or $z_0 \in \mathbb{R}$ and, consequently, $\lambda_0 \in \mathbb{R}$. Hence the following inequalities hold

$$\frac{1}{2}\left(k(z_0,\lambda_0)-1\right) \le \sharp(\Xi_1^+[z_0,\lambda_0]) \le \frac{1}{2}\left(k(z_0,\lambda_0)+1\right) \ . \tag{4.9}$$

In the case $k(z_0,\lambda_0) = 1$ the inequalities are obvious, in the case $z_0 \in \mathbb{R}$ they follow from Lemma 1.5.

(a) Let $\xi(\lambda) = \xi_j^+(\lambda)$, $n+1 \le j \le 2n$. By Lemma 1.1 the family of functions $\{\xi_r^+(\lambda)\}_{\substack{r=1\\ r\neq q}}^{2n}$ contains exactly $k\left(\xi_j^+(\lambda_0),\lambda_0\right)-1$ functions such that $\xi_r^+(\lambda_0) = \xi_j^+(\lambda_0) =: z_0$. Since $\theta_q(\lambda_0) \neq z_0$, the family $\{\theta_r^+(\lambda)\}_{\substack{r=1\\ r\neq q}}^{n}$ contains $\sharp(\Xi_1^+[\xi_j^+(\lambda_0),\lambda_0])$ functions such that $\theta_r^+(\lambda_0) = \xi_j^+(\lambda_0) = z_0$.

For all functions from Ξ taking the value z_0 in λ_0 inequality (4.5) holds true in a neighbourhood of λ_0. Therefore, for λ in a neighbourhood of λ_0 we have

$$\left|\frac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2}{p'\left(\xi_j^+(\lambda)\right)}\right| = \frac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2}{\prod\limits_{\substack{r=1\\ r\neq j}}^{2n}\left|\xi_r^+(\lambda)-\xi_j^+(\lambda)\right|} \le$$

$$\le C_1\frac{|\lambda-\lambda_0|^{\frac{2\sharp(\Xi_1^+[z_0,\lambda_0])}{k(z_0,\lambda_0)}}}{|\lambda-\lambda_0|^{\frac{k(z_0,\lambda_0)-1}{k(z_0,\lambda_0)}}} = C_1|\lambda-\lambda_0|^{\frac{2\sharp(\Xi_1^+[z_0,\lambda_0])-(k(z_0,\lambda_0)-1)}{k(z_0,\lambda_0)}} \quad ,$$

where C_1 is some constant. Inequality (4.9) implies $2\sharp(\Xi_1^+[z_0,\lambda_0]) \ge k(z_0,\lambda_0)-1$. This proves the statement.

(b) In the case $\lambda_0 = \infty$ the conclusion follows easily from Propositions 4.2 (d) and 4.3 (a). Let λ_0 is finite and $\theta_q = \xi_q^+ \in \Xi_1^+$. Denote $z_0 = \theta_q(\lambda_0)$. By Proposition 4.2 (b), $\theta_q(\lambda_0) \neq \theta_r(\lambda_0)$ for $r = n+1,\dots,2n$, as in this case $\theta_r \in \Xi^-$ and $\lambda_0 \neq 0$. There are exactly $\sharp(\Xi_1^+[z_0,\lambda_0])-1$ functions with the property $\theta_r(\lambda_0) = z_0 = \theta_q(\lambda_0)$ among the functions $\{\theta_r(\lambda)\}_{\substack{r=1\\ r\neq q}}^{n}$. Therefore, for λ from a neighbourhood of λ_0

$$\left|\frac{\operatorname{Im}\lambda}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}|\theta_r(\lambda)-\theta_q(\lambda)|^2\operatorname{Im}\theta_q(\lambda)}\right| \le C_1\frac{\operatorname{Im}\lambda}{|\lambda-\lambda_0|^{\frac{2\left(\sharp(\Xi_1^+[z_0,\lambda_0])-1\right)}{k(z_0,\lambda_0)}}|\operatorname{Im}\theta_q(\lambda)|} =$$

$$= C_1|\lambda-\lambda_0|^{\frac{k(z_0,\lambda_0)+1-2\sharp(\Xi_1^+[z_0,\lambda_0])}{k(z_0,\lambda_0)}}\frac{\operatorname{Im}\lambda}{|\lambda-\lambda_0|^{\frac{k(z_0,\lambda_0)-1}{k(z_0,\lambda_0)}}|\operatorname{Im}\theta_q(\lambda)|} \ ,$$

where C_1 is some constant. By inequality (4.9) $k(z_0,\lambda_0)+1 \geq 2\sharp(\Xi_1^+[z_0,\lambda_0])$. In the case $z_0=\theta_q(\lambda_0)\in\mathbb{R}$ we obtain the statement (b) from this and from Proposition 4.3 (b). In the case $z_0\notin\mathbb{R}$, $k(z_0,\lambda_0)=1$, the statement (b) is evident.

(c) We deal with the functions B_{jq}^+, $n+1\leq j\leq 2n$.

If $\lambda_0\notin\{0,\infty\}$ and $\theta_q(\lambda_0)\neq\xi_j^+(\lambda_0)$, then the statement (c) follows from (a) and (b).

Consider the case $\theta_q(\lambda_0)=\xi_j^+(\lambda_0)=z_0$, $\lambda_0\neq\infty$. Then the inequality $\dfrac{|\xi(\lambda)-\xi_j^+(\lambda)|}{|\xi(\lambda)-\theta_q(\lambda)|}<C_1$ holds for all $\xi(\lambda)\in\Xi\setminus\{\xi_j^+,\theta_q\}$ in some neighbourhood of λ_0, where C_1 is a constant. This inequality follows immediately from Proposition 4.2 (a),(c). Therefore, we get

$$
\begin{aligned}
|B_{jq}^+(\lambda)| &= \frac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}|\theta_r(\lambda)-\theta_q(\lambda)|^2}\,\frac{|\operatorname{Im}\lambda|}{\prod\limits_{\substack{r=1\\ r\neq j}}^{2n}\left|\xi_r^+(\lambda)-\xi_j^+(\lambda)\right|\,|\operatorname{Im}\theta_q(\lambda)|}\\
&\leq C_2\frac{|\operatorname{Im}\lambda|}{|\lambda-\lambda_0|^{\frac{k(z_0,\lambda_0)-1}{k(z_0,\lambda_0)}}\,|\operatorname{Im}\theta_q(\lambda)|},
\end{aligned}
$$

with a constant C_2. As $\theta_q(\lambda_0)=z_0$, Proposition 4.3 (b) implies that the function B_{jq}^+ is bounded in a neighbourhood of λ_0.

Consider the case $\lambda_0=\infty$. Then the following inequality holds with some constant C_1 for λ big enough

$$
\begin{aligned}
|B_{jq}^+(\lambda)| \leq C_1\frac{|\lambda-\lambda_0|^{\frac{2(2n-1)}{2n}}}{|\lambda-\lambda_0|^{\frac{2(2n-1)}{2n}}}\,\frac{|\operatorname{Im}\lambda|}{|\lambda-\lambda_0|^{\frac{2n-1}{2n}}\,|\operatorname{Im}\theta_q(\lambda)|} &=\\
= C_1\frac{|\operatorname{Im}\lambda|}{|\lambda-\lambda_0|^{\frac{2n-1}{2n}}\,|\operatorname{Im}\theta_q(\lambda)|}&\,.
\end{aligned}
$$

Now Proposition 4.3 proves the statement.

Consider the case $\lambda_0=0$, $\theta_q(0)=z_1\neq\xi_j^+(0)=z_0$. Here we use the assumption that the polynomial $p(t)$ is nonnegative. From this assumption it follows that any real root t_0 of polynomial $p(t)$ has an even multiplicity and, by Lemma 1.5 (a),

$$
\sharp(\Xi_1^+[t_0,0])=\sharp(\Xi_1^-[t_0,0])=\frac{1}{2}k(t_0,0)\,. \tag{4.10}
$$

It is not difficult to verify that $\sharp(\Theta[z_2,0])=\sharp(\Xi_1^+[z_2,0])+\sharp(\Xi_1^-[z_2,0])=k(z_2,0)$. Indeed, if $z_2\in\mathbb{R}$ it follows from (4.10). If $z_2\notin\mathbb{R}$, for example $z_2\in\mathbb{C}_+$, this fact follows from the equalities $\sharp(\Xi_1^+[z_2,0])=k(z_2,0)$, $\sharp(\Xi_1^-[z_2,0])=0$. Thus, there exist exactly $k(z_1,0)-1$ functions with the property $\theta_r(0)=\theta_q(0)=z_1$ among the functions $\{\theta_r(\lambda)\}_{\substack{r=1\\ r\neq q}}^{2n}$. Since $\xi_j^+(0)=z_0\neq\theta_q(0)$, the family $\{\theta_r(\lambda)\}_{\substack{r=1\\ r\neq q}}^{2n}$ contains exactly $k(z_0,0)$ functions such that $\theta_r(0)=\xi_j^+(0)=z_0$. Hence, for λ sufficiently close to zero one has

$$\left(\prod_{\substack{r=1\\r\neq q}}^{2n}|\theta_r(\lambda)-\theta_q(\lambda)|^2\right)^{-1} \leq C_1|\lambda|^{-\frac{2(k(z_1,0)-1)}{k(z_1,0)}},$$

$$\prod_{\substack{r=1\\r\neq q}}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2 \leq C_1|\lambda|^{\frac{2k(z_0,0)}{k(z_0,0)}}$$

with some constant C_1. Therefore

$$|B_{jq}^+(\lambda)| = \frac{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2}{\prod\limits_{\substack{r=1\\r\neq q}}^{2n}|\theta_r(\lambda)-\theta_q(\lambda)|^2}\frac{|\operatorname{Im}\lambda|}{\left|p'\left(\xi_j^+(\lambda)\right)\right||\operatorname{Im}\theta_q(\lambda)|} \leq$$

$$\leq C_2\frac{|\lambda|^{\frac{2k(z_0,0)}{k(z_0,0)}}}{|\lambda|^{\frac{2(k(z_1,0)-1)}{k(z_1,0)}}}\frac{|\operatorname{Im}\lambda|}{|\lambda|^{\frac{k(z_0,0)-1}{k(z_0,0)}}|\operatorname{Im}\theta_q(\lambda)|} =$$

$$= C_2|\lambda|^{\frac{1}{k(z_0,0)}+\frac{1}{k(z_1,0)}}\frac{|\operatorname{Im}\lambda|}{|\lambda|^{\frac{k(z_1,0)-1}{k(z_1,0)}}|\operatorname{Im}\theta_q(\lambda)|},$$

where C_2 is some constants.

Combining this inequality with Proposition 4.3 (b) one obtains the lemma. □

Remark 4.3. If the polynomial $p(t)$ changes sign, then some functions $B_{jq}^{\pm}$ are not bounded in any neighbourhood of zero.

Remark 4.4. The function $B_{jq}^{\pm}$ is not bounded in any neighbourhood of λ_0 if $\lambda_0 \in \left(\Lambda(\xi_j^{\pm})\cup\Lambda(\theta_q)\right)\setminus\{0\}$.

Now we get integral estimations of form (0.2),(0.3) for some groups of summands from (2.8),(2.9).

Let z_1 be an arbitrary root of one of the polynomials $p(z)-\lambda_1$, $p(z)+\lambda_1$. For such λ_1, z_1 we put

$$\Lambda^\delta(\lambda_0,z_1) := \{\lambda : |\xi(\lambda)-z_1|<\delta \quad \forall\xi(\lambda)\in\Xi[z_1,\lambda_0]\ \}\ .$$

We take δ such that the following statements holds true. From $\lambda_0\neq\lambda_1$, $\lambda_1\neq 0$, $k(z_0,\lambda_0)>1$, $k(z_1,\lambda_1)\geq 1$, $\{z_0,z_1\}\subset\mathbb{C}\setminus\mathbb{R}$ it follows that

$$\begin{gathered}\Lambda^\delta(\lambda_0,z_0)\cap\Lambda^\delta(\lambda_1,z_1)=\emptyset,\ 0\notin\overline{\Lambda^\delta(\lambda_1,z_1)},\\ \delta<\frac{1}{5}\min_{\xi\in\Xi\setminus\Xi[z_0,\lambda_0]}|z_0-\xi(\lambda_0)|,\ \delta<\frac{1}{3}\min_{z_0\in Z_0\setminus\mathbb{R}}|\operatorname{Im}z_0|\ .\end{gathered} \tag{4.11}$$

We can choose such δ, because the sets Z_0, Λ_0 are finite and the functions from the family Ξ are continuous. Note that the set $\Lambda^\delta(\lambda_0,z_0)$ is a bounded neighbourhood of λ_0.

We put

$$\Lambda^\delta(\lambda_0) := \bigcap_{z_0\,:\,\Xi[z_0,\lambda_0]\neq\emptyset} \Lambda^\delta(\lambda_0, z_0)\ ,$$

$$\Lambda^+_{jq} := \Lambda_+ \setminus \bigcup_{\lambda_0\in\Lambda(\xi_j^+)\cup\Lambda(\theta_q)\setminus\{0\}} \Lambda^\delta(\lambda_0)\ ,\ \Lambda^-_{jq} := \Lambda_+ \setminus \bigcup_{\lambda_0\in\Lambda(\xi_j^-)\cup\Lambda(\theta_q)\setminus\{0\}} \Lambda^\delta(\lambda_0)\ ,$$

for $j = n+1, ..., 2n$, $q = 1, ..., 2n$.

By $\int\limits_{\lambda\in U} g(\lambda)\, d\mu$ we understand the integral of the function $g(\mu + i\varepsilon)$ over the set of $\mu\in\mathbb{R}$ such that $\lambda = \mu + i\varepsilon \in U$, while ε is fixed. We denote the residue of a holomorphic function $g(z)$ in z_0 by $\operatorname*{res}_{z=z_0} g(z)$.

Lemma 4.2. *For all $\varepsilon > 0$ and $f \in L^2(\mathbb{R})$ the following statements are valid.*

(a) If the polynomial $p(t)$ is nonnegative, then

$$\varepsilon \int\limits_{\lambda\in\Lambda^\pm_{jq}} \|y^\pm_{jq}(x,\lambda)\|^2_{L^2} d\mu \le m^\pm_{jq}\|f_\pm(x)\|^2_{L^2}.$$

(b) Let $z_0 \in Z_0\setminus\mathbb{R}$, $k(z_0, \pm\lambda_0) > 1$, $\lambda_0 \neq 0$, $\lambda_0 \notin \Lambda(\theta_q)$, $1\le q\le 2n$, $\Xi^\pm_2[z_0,\lambda_0]\neq\emptyset$. Then

$$\varepsilon \int\limits_{\lambda\in\Lambda^\delta(\lambda_0,z_0)} \left\| \sum_{\xi_j^\pm\in\Xi_2^\pm[z_0,\lambda_0]} y^\pm_{jq}(x,\lambda)\right\|^2_{L^2} d\mu \le m^\pm_q(\lambda_0,z_0)\|f_\pm(x)\|^2_{L^2}.$$

(c) Let $z_0\in Z_0\setminus\mathbb{R}$, $\lambda_0\neq 0$, $\lambda_0\notin\Lambda(\xi_j^\pm)$, $n+1\le j\le 2n$. Let $k(z_0,\lambda_0) > 1$ or $k(z_0,-\lambda_0) > 1$. Then

$$\varepsilon \int\limits_{\lambda\in\Lambda^\delta(\lambda_0,z_0)} \left\| \sum_{\theta_q\in\Theta[z_0,\lambda_0]} y^\pm_{jq}(x,\lambda)\right\|^2_{L^2} d\mu \le m^\pm_j(\lambda_0,z_0)\|f_\pm(x)\|^2_{L^2}.$$

(d) Let $\{z_0,z_1\}\in Z_0\setminus\mathbb{R}$, $\lambda_0\neq 0$, $k(z_0,\pm\lambda_0) > 1$, $\Xi^\pm_2[z_0,\lambda_0]\neq\emptyset$. Let $k(z_1,\lambda_0) > 1$ or $k(z_1,-\lambda_0) > 1$. Then

$$\varepsilon \int\limits_{\lambda\in\Lambda^\delta(\lambda_0)} \left\| \sum_{\xi_j^\pm\in\Xi_2^\pm[z_0,\lambda_0]}\ \sum_{\theta_q\in\Theta[z_0,\lambda_0]} y^\pm_{jq}(x,\lambda)\right\|^2_{L^2} d\mu \le m^\pm(\lambda_0,z_0,z_1)\|f_\pm(x)\|^2_{L^2}.$$

Here $m^\pm_{jq}$, $m^\pm_q(\lambda_0,z_0)$, $m^\pm_j(\lambda_0,z_0)$, $m(\lambda_0,z_0,z_1)$ are some constants independent of ε and f.

Proof. Let $\Lambda_+ = \{\lambda\in\mathbb{C} : \varepsilon_N < \operatorname{Im}\lambda < \varepsilon_{N+1}\}$ be one of the strips Λ^0_+, Λ^1_+, ..., Λ^{2n-1}_+. We prove the statements for $\varepsilon\in(\varepsilon_N,\varepsilon_{N+1}]$, $0\le N\le 2n-1$. Then we combine estimations for all the strips and obtain the proof of the lemma. It can be done because the number of strips is finite. So we suppose henceforth that $\varepsilon\in[\varepsilon_N,\varepsilon_{N+1}]$ and $\lambda\in\overline{\Lambda_+}$.

(a) We prove the statement for y^+_{jq}, $n+1\le j\le 2n$. For y^-_{jq}, $n+1\le j\le 2n$ the proof is the same.

As $\overline{\Lambda^+_{jq}} \subset \{0,\infty\}\cup\left(\overline{\Lambda_+}\setminus\left(\Lambda(\xi_j^+)\cap\Lambda(\theta_q)\right)\right)$ and, by conditions of the lemma, the polynomial $p(t)$ is nonnegative, we conclude from Lemma 4.1 (c) that the function $B^+_{jq}(\lambda)$ is bounded in a neighbourhood of each point of the set $\overline{\Lambda^+_{jq}}$. Since the set $\overline{\Lambda^+_{jq}}$ is compact in the topology of $\overline{\mathbb{C}}$, the function $B^+_{jq}(\lambda)$ is bounded on $\overline{\Lambda^+_{jq}}$, $\max\limits_{\lambda\in\overline{\Lambda^+_{jq}}} B^+_{jq}(\lambda) < \infty$. From this and from inequality 4.3 it follows statement (a).

(b) Let $z_0 \in Z_0 \cap \mathbb{C}_-$, $k(z_0,\lambda_0) > 1$, $\lambda_0 \neq 0$, $\lambda_0 \notin \Lambda(\theta_q)$. Then $\Xi[z_0,\lambda_0] = \Xi^+[z_0,\lambda_0] = \Xi^+_2[z_0,\lambda_0] \neq \emptyset$, $\Xi^-_2[z_0,\lambda_0] = \emptyset$. The remaining cases may be considered similarly.

Let $\lambda \in \Lambda^\delta(\lambda_0, z_0)$. Consider the sum (see [12, I.I.25])

$$\sum_{\xi^+_j\in\Xi^+_2[z_0,\lambda_0]} \widehat{y^+_{jq}}(t) = -\sum_{\xi^+_j\in\Xi^+_2[z_0,\lambda_0]} \frac{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \xi^+_j)}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \theta_q)} \frac{\widehat{f_+}(\xi^+_j)}{p'(\xi^+_j)} \frac{1}{\theta_q - t} =$$

$$= -\frac{1}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \theta_q)(\theta_q - t)} \sum_{\xi^+_j\in\Xi^+_2[z_0,\lambda_0]} \frac{\prod\limits_{r=1}^{2n}(\theta_r - \xi^+_j)\widehat{f_+}(\xi^+_j)}{p'(\xi^+_j)(\theta_q - \xi^+_j)} =$$

$$= -\frac{1}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r - \theta_q)(\theta_q - t)} \sum_{\xi^+_j\in\Xi^+_2[z_0,\lambda_0]} \operatorname*{res}_{w=\xi^+_j(\lambda)} \frac{\prod\limits_{r=1}^{2n}(\theta_r - w)\widehat{f_+}(w)}{(p(w)-\lambda)(\theta_q - w)} .$$

By virtue of (4.11), only points $\{\xi^+_j(\lambda) : \xi^+_j \in \Xi^+_2[z_0,\lambda_0]\}$ may be singular points of the function $\dfrac{\prod\limits_{r=1}^{2n}(\theta_r - w)\widehat{f_+}(w)}{(p(w)-\lambda)(\theta_q - w)}$ in the domain $\{w : |w - z_0| < 2\delta\} \subset \mathbb{C}_-$. Therefore

$$\sum_{\xi^+_j\in\Xi^+_2[z_0,\lambda_0]} \widehat{y^+_{jq}}(t,\lambda) =$$

$$= -\frac{1}{2\pi i} \frac{1}{\prod\limits_{\substack{r=1\\ r\neq q}}^{2n}(\theta_r(\lambda) - \theta_q(\lambda))(\theta_q(\lambda) - t)} \oint\limits_{|w-z_0|=2\delta} \frac{\prod\limits_{r=1}^{2n}(\theta_r(\lambda) - w)\widehat{f_+}(w)}{(p(w)-\lambda)(\theta_q(\lambda) - w)} dw ,$$

here $\oint$ denotes integral taken over one cycle.

Let $\lambda \in \Lambda^\delta(\lambda_0, z_0)$ and $|w_0 - z_0| = 2\delta$. Then, by virtue of (4.11), $|\xi(\lambda) - w_0| > 2\delta$ for

$\xi \notin \Xi[z_0, \lambda_0]$ and $|\xi(\lambda) - w_0| > \delta$ for $\xi \in \Xi[z_0, \lambda_0]$. Therefore

$$C_1 := \max_{\substack{\lambda \in \Lambda^\delta(\lambda_0, z_0) \\ w: |w - z_0| = 2\delta}} \frac{\prod_{r=1}^{2n} |\theta_r(\lambda) - w|}{|\theta_q(\lambda) - w||p(w) - \lambda|} =$$

$$= \max_{\substack{\lambda \in \Lambda^\delta(\lambda_0, z_0) \\ w: |w - z_0| = 2\delta}} \frac{\prod_{r=1}^{2n} |\theta_r(\lambda) - w|}{|\theta_q(\lambda) - w| \prod_{r=1}^{2n} |\xi_r^+(\lambda) - w|} < \infty\,.$$

Moreover, since $\widehat{f_+}(w) \in H^2(\mathbb{C}_-)$ and, by virtue of (4.11), $w_0 \in \mathbb{C}_-$, $\operatorname{Im} w_0 < -\delta$, we have $|f_+(w_0)| \leq |\operatorname{Im} w_0|^{-1/2} \|\widehat{f_+}(w)\|_{H^2} \leq \delta^{-1/2} \|f_+(x)\|_{L^2}$ (see [8, vi. C]). Combining these estimations, we get

$$\left| \oint\limits_{|w - z_0| = 2\delta} \frac{\prod_{r=1}^{2n} (\theta_r(\lambda) - w) \widehat{f_+}(w)}{(p(w) - \lambda)(\theta_q(\lambda) - w)} dw \right| \leq$$

$$\leq \oint\limits_{|w - z_0| = 2\delta} \frac{\prod_{r=1}^{2n} |\theta_r(\lambda) - w|}{|p(w) - \lambda||\theta_q(\lambda) - w|} \delta^{-1/2} \|f_+(x)\|_{L^2} |dw| \leq$$

$$\leq \delta^{-1/2} \|f_+(x)\|_{L^2} C_1 \oint\limits_{|w - z_0| = 2\delta} |dw| = 4C_1 \pi \delta^{1/2} \|f_+(x)\|_{L^2}.$$

Hence,

$$\varepsilon \int\limits_{\lambda \in \Lambda^\delta(\lambda_0, z_0)} \left\| \sum_{\xi_j^+ \in \Xi_2^+[z_0, \lambda_0]} y_{jq}^+(x, \lambda) \right\|_{L^2}^2 d\mu \leq$$

$$\leq \varepsilon \int\limits_{\lambda \in \Lambda^\delta(\lambda_0, z_0)} 4\delta C_1^2 \|f_+(x)\|_{L^2}^2 \frac{1}{\prod_{\substack{r=1 \\ r \neq q}}^{2n} |\theta_r(\lambda) - \theta_q(\lambda)|^2} \left\| \frac{1}{\theta_q(\lambda) - t} \right\|_{L^2}^2 d\mu \leq$$

$$\leq 4\delta C_1^2 \|f_+(x)\|_{L^2}^2 \int\limits_{\lambda \in \Lambda^\delta(\lambda_0, z_0)} \frac{\operatorname{Im} \lambda}{\prod_{\substack{r=1 \\ r \neq q}}^{2n} |\theta_r(\lambda) - \theta_q(\lambda)|^2 \; 2|\operatorname{Im} \theta_q(\lambda)|} d\mu\;.$$

According to (4.11) the sets $\overline{\Lambda^\delta(\lambda_0, z_0)}$ and $\Lambda(\theta_q) \cup \{0\}$ are disjoint, $\overline{\Lambda^\delta(\lambda_0, z_0)} \cap (\Lambda(\theta_q) \cup \{0\}) = \emptyset$, as $\lambda_0 \neq 0$ and $\lambda_0 \notin \Lambda(\theta_q)$. Then, by Lemma 4.1, the function $\dfrac{\operatorname{Im} \lambda}{\prod_{\substack{r=1 \\ r \neq q}}^{2n} |\theta_r(\lambda) - \theta_q(\lambda)|^2 |\operatorname{Im} \theta_q(\lambda)|}$ is bounded in a neighbourhood of each point of the set $\overline{\Lambda^\delta(\lambda_0, z_0)} \cap \Lambda_+$. Consequently, this function is bounded on the set $\overline{\Lambda^\delta(\lambda_0, z_0)} \cap \Lambda_+$. This proves statement (b), since the set $\Lambda^\delta(\lambda_0, z_0)$ is bounded.

(c) Let $z_0 \in Z_0 \cap \mathbb{C}_+$, $k(z_0, \lambda_0) > 1$, $\lambda_0 \neq 0$, $\lambda_0 \notin \Lambda(\xi_j^+)$, $n+1 \leq j \leq 2n$. Then $\Xi[z_0, \lambda_0] = \Theta[z_0, \lambda_0] = \Xi_1^+[z_0, \lambda_0] \neq \emptyset$, $\Xi_1^-[z_0, \lambda_0] = \emptyset$. The remaining cases may be considered similarly.

Let $\lambda \in \Lambda^\delta(\lambda_0, z_0)$. Consider the sum

$$\sum_{\theta_q \in \Theta[z_0,\lambda_0]} \widehat{y_{jq}^+}(t) =$$

$$= -\frac{\prod_{r=1}^{2n}(\theta_r - \xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)} \sum_{\theta_q \in \Xi_1^+[z_0,\lambda_0]} \operatorname*{res}_{z=\theta_q(\lambda)} \frac{1}{\prod_{r=1}^{2n}(\theta_r - z)(z - \xi_j^+)(z-t)} =$$

$$= -\frac{1}{2\pi i} \frac{\prod_{r=1}^{2n}(\theta_r(\lambda) - \xi_j^+(\lambda))\widehat{f_+}(\xi_j^+(\lambda))}{p'(\xi_j^+(\lambda))} \oint_{|z-z_0|=2\delta} \frac{1}{\prod_{r=1}^{2n}(\theta_r(\lambda) - z)(z - \xi_j^+(\lambda))} \frac{1}{z-t} dz\ . \tag{4.12}$$

The last equality holds true, because only points $\{\theta_q(\lambda) : \theta_q \in \Xi_1^+[z_0, \lambda_0]\}$ may be singular points of the integrand function in the domain $\{z : |z - z_0| < 2\delta\}$.

By virtue of (4.11)

$$C_1 := \max_{\substack{\lambda \in \Lambda^\delta(\lambda_0, z_0) \\ z: |z-z_0|=2\delta}} \frac{1}{\prod_{r=1}^{2n} |\theta_r(\lambda) - z||z - \xi_j^+(\lambda)|} < \infty\ .$$

Moreover, if $|z - z_0| = 2\delta$ then, by virtue of (4.11), $\operatorname{Im} z > \delta$. Therefore

$$\left\| \oint_{|z-z_0|=2\delta} \frac{1}{\prod_{r=1}^{2n}(\theta_r(\lambda) - z)(z - \xi_j^+(\lambda))} \frac{1}{z-t} dz \right\|_{L^2} \leq$$

$$\leq \oint_{|z-z_0|=2\delta} \frac{1}{\prod_{r=1}^{2n} |\theta_r(\lambda) - z||z - \xi_j^+(\lambda)|} \left\| \frac{1}{z-t} \right\|_{L^2} |dz| \leq$$

$$\leq C_1 \oint_{|z-z_0|=2\delta} \frac{1}{2|\operatorname{Im} z|^{1/2}} |dz| \leq 2\pi C_1 \delta^{1/2}\ . \tag{4.13}$$

From $\lambda_0 \notin \Lambda(\xi_j^+)$ it follows $\overline{\Lambda^\delta(\lambda_0, z_0)} \cap \Lambda(\xi_j^+) = \emptyset$. Taking Lemma 4.1 (a) into account, one has

$$C_2 := \sup_{\lambda \in \Lambda^\delta(\lambda_0, z_0) \cap \Lambda_+} \frac{\prod_{r=1}^{2n} |\theta_r(\lambda) - \xi_j^+(\lambda)|^2}{|p'(\xi_j^+(\lambda))|} < \infty.$$

Furthermore, the set $\Lambda^\delta(\lambda_0, z_0)$ is bounded, hence $C_3 := \sup_{\lambda \in \Lambda^\delta(\lambda_0, z_0)} \varepsilon = \sup_{\lambda \in \Lambda^\delta(\lambda_0, z_0)} \operatorname{Im} \lambda < \infty$.

Therefore, combining (4.12) and (4.13), one obtains

$$\varepsilon \int\limits_{\lambda\in\Lambda^\delta(\lambda_0,z_0)} \left\| \sum_{\theta_q\in\Theta[z_0,\lambda_0]} y^+_{jq}(x,\lambda)\right\|^2_{L^2} d\mu \leq$$

$$\leq C_1^2 C_3 \delta \int\limits_{\lambda\in\Lambda^\delta(\lambda_0,z_0)} \frac{\prod\limits_{r=1}^{2n}\left|\theta_r(\lambda)-\xi_j^+(\lambda)\right|^2}{\left|p'\left(\xi_j^+(\lambda)\right)\right|} \left|\widehat{f_+}\left(\xi_j^+(\lambda)\right)\right|^2 \left|\frac{d\xi_j^+(\mu+i\varepsilon)}{d\mu}\right| d\mu \leq$$

$$\leq C_1^2 C_2 C_3 \delta \int\limits_{\mu:\lambda\in\Lambda^\delta(\lambda_0,z_0)} \left|\widehat{f_+}\left(\xi_j^+(\lambda)\right)\right|^2 \left|d\xi_j^+(\mu+i\varepsilon)\right| .$$

Applying Lemma 1.6, we finish the proof of the statement.

(d) Under the conditions of the lemma we have $z_0 \neq z_1$. Indeed, $\lambda_0 \neq 0$ and, by Proposition 4.2 (b),

$$\left\{z : \Xi_2^{\pm}[z,\lambda_0] \neq \emptyset\right\} \cap \left\{z : \Xi_1^{\mp}[z,\lambda_0] \neq \emptyset\right\} = \emptyset.$$

If $z \notin \mathbb{R}$ and $\Xi_2^{\pm}[z,\lambda_0] \neq \emptyset$, then $\Xi_1^{\pm}[z,\lambda_0] = \emptyset$, as values of the functions from $\Xi_2^{\pm}$ and $\Xi_1^{\pm}$ belong to different half-planes. Therefore

$$\left\{z : \Xi_2^{\pm}[z,\lambda_0] \neq \emptyset\right\} \cap \left\{z : \Theta[z,\lambda_0] \neq \emptyset\right\} = \emptyset.$$

Since $z_0 \in \{z : \Xi_2^{\pm}[z,\lambda_0] \neq \emptyset\}$ and $z_1 \in \{z : \Theta[z,\lambda_0] \neq \emptyset\}$, we conclude that $z_0 \neq z_1$.

Let $z_0 \in Z_0 \cap \mathbb{C}_-$, $z_1 \in Z_0 \cap \mathbb{C}_+$. The remaining cases may be considered similarly. Then $\Xi[z_0,\lambda_0] = \Xi_2^+[z_0,\lambda_0]$, $\Xi[z_1,\lambda_0] = \Xi_1^+[z_1,\lambda_0]$.

Let $\lambda \in \Lambda^\delta(\lambda_0)$, $\lambda \neq \lambda_0$. Then $\lambda \in \Lambda^\delta(\lambda_0,z_0) \cap \Lambda^\delta(\lambda_0,z_1)$. Consider the sum

$$\sum_{\xi_j^+\in\Xi_2^+[z_0,\lambda_0]} \sum_{\theta_q\in\Theta[z_0,\lambda_0]} \widehat{y^+_{jq}}(t) =$$

$$= -\sum_{\xi_j^+\in\Xi_2^+[z_0,\lambda_0]} \frac{\prod\limits_{r=1}^{2n}(\theta_r-\xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)} \sum_{\theta_q\in\Xi_1^+[z_0,\lambda_0]} \operatorname*{res}_{z=\theta_q(\lambda)} \frac{1}{\prod\limits_{r=1}^{2n}(\theta_r-z)(z-\xi_j^+)(z-t)} =$$

$$= -\frac{1}{2\pi i}\oint\limits_{|z-z_1|=2\delta} \frac{1}{\prod\limits_{r=1}^{2n}(\theta_r-z)}\frac{1}{z-t} \sum_{\xi_j^+\in\Xi_2^+[z_0,\lambda_0]} \frac{\prod\limits_{r=1}^{2n}(\theta_r-\xi_j^+)\widehat{f_+}(\xi_j^+)}{p'(\xi_j^+)(z-\xi_j^+)} dz =$$

$$= -\frac{1}{2\pi i}\oint\limits_{|z-z_1|=2\delta} \frac{1}{\prod\limits_{r=1}^{2n}(\theta_r-z)}\frac{1}{z-t} \sum_{\xi_j^+\in\Xi_2^+[z_0,\lambda_0]} \operatorname*{res}_{w=\xi_j^+(\lambda)} \frac{\prod\limits_{r=1}^{2n}(\theta_r-w)\widehat{f_+}(w)}{(p(w)-\lambda)(z-w)} dz =$$

$$= \frac{1}{4\pi^2}\oint\limits_{|z-z_1|=2\delta} \frac{1}{\prod\limits_{r=1}^{2n}(\theta_r(\lambda)-z)}\frac{1}{z-t} \oint\limits_{|w-z_0|=2\delta} \frac{\prod\limits_{r=1}^{2n}(\theta_r(\lambda)-w)\widehat{f_+}(w)}{(p(w)-\lambda)(z-w)} dw\ dz .$$

Here we use the fact that, by virtue of (4.11), the circles $\{z : |z-z_1| = 2\delta\}$ and $\{z : |z-z_0| = 2\delta\}$ are disjoint, $\min\limits_{\substack{z:|z-z_1|=2\delta \\ w:|w-z_0|=2\delta}} |z-w| > \delta$. Therefore, the function $\dfrac{1}{z-w}$ is holomorphic in the circle $\{w : |w-z_0| \le 2\delta\}$.

Arguing as in the proofs of (b) and (c), we conclude that $\lambda \in \Lambda^{\delta}(\lambda_0) \cap \Lambda_+$ implies

$$\left| \oint\limits_{|w-z_0|=2\delta} \frac{\prod\limits_{r=1}^{2n} (\theta_r(\lambda) - w)\widehat{f_+}(w)}{(p(w)-\lambda)(z-w)} dw \right| \le C_4 \|f_+(x)\|_{L^2}$$

for $z : |z-z_1| = 2\delta$ and

$$\oint\limits_{|z-z_1|=2\delta} \frac{1}{\prod\limits_{r=1}^{2n} |\theta_r(\lambda) - z|} \left\| \frac{1}{z-t} \right\|_{L^2} |dz| \le C_5 \ ,$$

where C_4, C_5 are some constants. The combination of these estimates proves (d). □

Let $p(t)$ be a nonnegative polynomial. Combining the estimates from Lemma 4.2 we get

$$\varepsilon \int\limits_{-\infty}^{\infty} \|y_0^+(x,\lambda)\|_{L^2}^2 d\mu = \varepsilon \int\limits_{-\infty}^{\infty} \|(y_0^+\chi_+ - y_0^+\chi_-)(x,\lambda)\|_{L^2}^2 d\mu \le m_0^+ \|f_+(x)\|_{L^2}^2,$$

$$\varepsilon \int\limits_{-\infty}^{\infty} \|y_0^-(x,\lambda)\|_{L^2}^2 d\mu = \varepsilon \int\limits_{-\infty}^{\infty} \|(y_0^-\chi_+ - y_0^-\chi_-)(x,\lambda)\|_{L^2}^2 d\mu \le m_0^- \|f_-(x)\|_{L^2}^2.$$

for all $\varepsilon > 0$ and for all $f \in L^2(\mathbb{R})$, where m_0^+, m_0^- are some constants.

Taking (4.1) and (4.2) into account, we get inequalities (0.2),(0.3) with some constants m_+, m_+^* independent of ε and f.

Thus, it follows from Theorem 0.2 that *if the polynomial $p(t)$ is nonnegative then the operator $A = Jp(D)$ is similar to a selfadjoint one.* Remembering Theorem 3.1 we obtain Theorem 0.1.

Corollary 4.1. *The operator* $\operatorname{sgn}(x-x_0)\, p(D)$ *is similar to a selfadjoint one if and only if the polynomial $p(t)$ is nonnegative.*

Proof. Let S_{x_0} be a translation operator, $(S_{x_0}f)(x) = f(x-x_0)$. Note that $S_{-x_0} = S_{x_0}^{-1} = S_{x_0}^*$. Then $\operatorname{sgn}(x-x_0)\, p(D) = S_{x_0} J S_{x_0}^{-1} p(D) = S_{x_0} J p(D) S_{x_0}^{-1}$. Hence the operators $\operatorname{sgn}(x-x_0)\, p(D)$ and $\operatorname{sgn} x\, p(D)$ are similar. Therefore, the corollary follows at once from Theorem 0.1. □

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