# THE WIENER TEST FOR QUASILINEAR ELLIPTIC EQUATIONS WITH NON - STANDARD GROWTH CONDITIONS

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Let $D$ be a bounded domain in $\mathbb{R}^n$. Consider in $D$ the quasilinear partial dirrefential equation

$$Lu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) = 0 \tag{1}$$

where $p(x)$ is a measurable in $D$ function and

$$1 < p_1 \leq p(x) \leq p_2 < \infty.$$

For strict definition of solution for equation (1) we introduce some classes of functions. Let

$$V\,(D) = \left\{ \psi(x) : \ \psi \in W_1^1\,(D)\,, \ |\nabla \psi|^p \in L_1\,(D) \right\},$$

where by $W_1^1\,(D)$ denote the classical Sobolev space with the norm

$$\| \ u \ \|_{W_1^1(D)} = \int\limits_D (|u| + |\nabla u|) \ dx.$$

Under the class $V_0\,(D)$ we shall understood the subset of $V\,(D)$ such that for any $u \in V_0\,(D)$ exists a sequence of functions $u_j \in V\,(D)$ with compact supports in $D$ satisfying relations

$$\lim_{j\to\infty} \|u_j - u\|_{W_1^1(D)} = 0, \ \lim_{j\to\infty} \int\limits_D |\nabla u_j|^p \ dx = \int\limits_D |\nabla u|^p \ dx. \tag{2}$$

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A function $u \in V(D)$ we shall call a solution of equation (1) if for every test function $\psi \in V_0(D)$ realized the integral identity

$$\sum_{i=1}^{n} \int_D |\nabla u|^{p(x)-2} u_{x_i} \psi_{x_i} \, dx = 0.$$

The solution of class $V_{loc}(D)$ may be define analogously.

The important question about density of smooth functions in $V(D)$ was investigated by Zhikov [1-4]. He proved that under assumption

$$|p(x) - p(y)| \leq \frac{const}{\ln \frac{1}{|x-y|}}, \; |x-y| \leq 1/2. \tag{3}$$

for any $u \in V(D)$ exists a sequence $u_j \in C^\infty(D)$ such that (2) holds.

The condition (3) is exact for this assertion. As showes the countrexample [3] the previous is false for $p(x)$ having the modulus of continuity $|\ln t|^{\varepsilon-1}$ for any $\varepsilon \in (0,1)$.

From countrexample [3] it follows that the solution of equation (1) may be not Hölder continuous in $D$ without condition (3). This result stimulated investigation of Hölder continuity for solutions (1).

The next result was obtained by Xianling Fan [5] and author of present paper [6] by different methods.

**Theorem 1.** *If condition (3) satisfies then any solution $u \in V_{loc}(D)$ of the equation (1) is Hölder continuous in any compact subset of $D$.*

The proof in [6] based on Trudinger's weak Harnack inequality [7]. We shall formulate it for supersolutions of equation (1): such functions $u$ that $Lu \leq 0$ in generalized sence.

Further will be make use of standard notation $B_r^{x_0}$ for open ball with radius $r$ and center $x_0$, $p_0 = p(x_0)$.

**Theorem 2.** *(Weak Harnack inequality.) Let $u \in V(B_{4r}^{x_0})$ be a nonnegative bounded supersolution of equaiton (1) in $B_{4r}^{x_0}$ and condition (3) satisfies. If $p_0 \leq n$ and $q > 0$ such that $q(n-p_0) < n(p_0-1)$ then for sufficiently small $r \leq r_0(n,p)$*

$$\left( \int_{B_{2r}^{x_0}} u^q \, dx \right)^{1/q} \leq c(n,p,q,M) r^{n/q} \left( \inf_{B_r^{x_0}} u + r \right),$$

*where $M = \sup\limits_{B_{4r}^{x_0}} u$.*

The weak Harnack inequality for supersolutions allowes to investigate a boundary behavior of solutions of the Dirichlet problem. We shall return to this question just a little later.

Now consider equation (1) with piecewise continuous function $p(x)$.

**Theorem 3.** *Let $D_1$ and $D_2$ are open subsets of $D$ with common Lipschitz boundary $\Sigma$ and $\overline{D} = \overline{D}_1 \cup \overline{D}_1$. If condition (3) satisfies in every $D_i$, $i = 1,2$, and $p(x)$ have nonzero jump on $\Sigma$ then any solution $u \in V_{loc}(D)$ of the equation (1) is Hölder continuous in any compact subset of $D$.*

The piecewise constant exponents $p(x)$ was investigated by Acerbi and Fusco [8].

Consider a question about the continuity at a boundary point $x_0 \in \partial D$ of solutions of the equation (1). At first using the construction of Kondratiev and Landis [9] define Wiener's generalized solution of the problem

$$Lu_f = 0 \ in \ D, \ u_f|_{\partial D} = f \tag{4}$$

with continuous on the boundary $\partial D$ function $f$.

The construction is based on the maximum principle and the solvability of the Dirichlet problem

$$Lu = 0 \ in \ D, \ (u - h) \in V_0(D), \ h \in V(D). \tag{5}$$

The function $u \in V(D)$ satisfying the equation (1) in the sence of integral identity and the boundary condition $(u - h) \in V_0(D)$ is called the solution of the Dirichlet problem (5). Unique solvability of this problem follows from the results of Zhikov [4]. The proof based on the fact that integral identity is the Euler equation for the corresponding variational problem.

Before to formulate a maximum principle we introduce the next notion. We shall say that the function $v \in V(D)$ is nonnegative in the sence of $V(D)$ (notation: $v \geq 0$) on compact subset $E \subset \overline{D}$, if for function $u = \inf(v, 0)$ exists a sequence $u_j \in V(D)$ such that $u_j = 0$ in a neghborhood of $\overline{D} \cap E$ and holds (2). If $u, v \in V(D)$ and $u - v \geq 0$ on $E$ in the sence of $V(D)$ we shall say that $u \geq v$ on $E$ in the sence of $V(D)$.

**Maximum principle.** *If $u$ and $v$ are two solutions belonging to $V(D)$ of the equation (1) in $D$ and $u \geq v$ on $\partial D$ in the sence of $V(D)$, then $u \geq v$ almost everywhere in $D$.*

For construction of the Wiener solution for the Dirichlet problem (4) we shall continue the boundary function $f$ on $\mathbb{R}^n$ continuously. Continued function as before denote by $f$. By $\{f_j\}$ denote a sequence of infinitely differentiable functions such that restrictions of $\{f_j\}$ on $\overline{D}$ converges uniformly to $f$ in $D$. Let us solve the Dirichlet problem

$$Lu_j = 0 \ in \ D, \ (u_j - f_j) \in V_0(D).$$

By maximum principle the sequence $\{u_j\}$ converges uniformly in compact subsets of the domain $D$ to some function $u_f$. This function does not depend on the methods of approximation and continuation of $f$ and is called the Wiener solution of the Dirichlet problem (4). It is not difficult to show that $u_f \in V_{loc}(D)$ satisfies equation (1). If $h \in V(D) \cap C(\overline{D})$ then the Wiener solution $u_f \in V_{loc}(D)$ of the problem (4) with the boundary function $f = h|_{\partial D}$ coincides with the solution of the problem (5).

**Definition 1.** *The boundary point $x_0 \in \partial D$ is called regular if for any continuous on $\partial D$ function $f$ the Wiener solution $u_f$ of the problem (4) is continuous at $x_0$.*

The criterion of regularity of a boundary point for Laplace equation was proved by Wiener [10]. This criterion is characterized by so call Wiener test. In the fundamental work Littman, Stampacchia, and Weinberger [11] showed that the same Wiener test identifies the regular boundary points whenever a uniformly elliptic linear operator with bounded measurable coefficients. The sufficient condition of regularity of the boundary point for $p$ - Laplace equation (equation (1) with $p = const$) was estsblished by Maz'ya [12]. He also received the estimate of modulus of continuity for solution near a regular boundary point. Later Gariepy and Ziemer [13] extended this result to a very general equation. For these equations some necessary condition of reqularity close to sufficient one was proved by Skrypnik [13]. The criterion of regularity of a boundary point for $p$ - Laplace equation was obtained by Kilpeläinen and Malý [14].

Let us define a notion of $V_p$ - capasity. Further we assume that $p(x) = p(x_0) = p_0$ in $\mathbb{R}^n \setminus D$.

**Definition 2.** *Let $E$ be a compact subset of $B_r$. The number*

$$C_p\left(E, B_r\right) = \inf \int\limits_{B_r} |\nabla \psi|^{p(x)} \, dx,$$

*where $\psi$ runs through all $\psi \in V_0\left(R_r\right)$ with $\psi \geq 1$ on $E$ in the sence of $V\left(R_r\right)$ is called $V_p$ - capasity of the set $E$ with respect to $B_r$.*

Put

$$\gamma_V(t) = C_p\left(\bar{B}_t^{x_0} \setminus D, B_{2t}^{x_0}\right) t^{p_0 - n}.$$

**Theorem 4.** *If condition (3) satisfies and $p_0 \leq n$ then for regularity of a boundary point $x_0 \in \partial D$ it is necessary and sufficiently to have*

$$\int\limits_0 \left[\gamma_V(t)\right]^{1/(p_0-1)} t^{-1} \, dt = \infty. \tag{6}$$

Let us give the estimate of modulus of continuity for solution (5) near a boundary point $x_0 \in \partial D$.

**Theorem 5.** *Let condition (3) satisfies and $u_f$ be the Wiener solution of the Dirichlet problem (5). Then for $\rho \leq \rho_0(n,p)$, $r \leq \rho/4$*

$$\underset{D \cap B_r^{x_0}}{osc} u_f \leq c \underset{D \cap B_\rho^{x_0}}{osc} f + c \underset{\partial D}{osc} f \exp\left(-\theta \int\limits_r^\rho \left[\gamma_V(t)\right]^{1/(p_0-1)} t^{-1} \, dt\right),$$

*if $p_0 \leq n$, or*

$$\underset{D \cap B_r^{x_0}}{osc} u_f \leq c \underset{D \cap B_\rho^{x_0}}{osc} f + c \underset{\partial D}{osc} f \left(r/\rho\right)^{1 - n/p_0},$$

*if $p_0 > n$. Here $c$ and $\theta$ are positive constants dependent only on n,p and* $\max\limits_{\partial D} |f|$.

Let us formulate a geometric conditions of regularity of a boundary point. We shall assume that $x_0 \in \partial D$ is coincides with the origin $O$ and the exterior of $D$ in the neighborhood of $O$ contain the domain

$$\left\{ 0 < x_n < a, \ \sum_{i=j+1}^{n-1} x_i^2 < g^2(x_n), \ |x_i| < a, \ i = 1, ..., j \right\},$$

where $g(t)$ is a continuous increasing function such that $t^\alpha < g(t) < t$.

**Theorem 6.** *The condition (6) is satisfied if*

$$\int\limits_0 \left(\frac{g(t)}{t}\right)^{\frac{n-1-j-p_0}{p_0-1}} t^{-1} \, dt = \infty,$$

*for* $p_0 < n-1-j$*, and if*

$$\int_0 |\ln g(t)|^{-1} t^{-1}\, dt = \infty,$$

*for* $p_0 = n-1-j$*. In the case* $p_0 > n-1-j$ *condition (6) is always satisfied.*

Earlier the analogous result for $p$ - Laplace equation was proved in [12].

**Theorem 7.** *Let condition (3) satisfies and* $f$ *be a Hölder continuous at* $x_0 \in \partial D$*. If the exterior of* $D$ *contain a cone with the vertex at* $x_0$ *then the generalized by Wiener solution of the Dirichlet problem (5) is Hölder continuous at* $x_0$*.*

All results of the present paper are correct for equations

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left( a_{ij}(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_j} \right) = 0,$$

where $a_{ij}(x)$ are measurable and bounded in $D$ functions such that for $x \in D$, $\xi \in \mathbb{R}^n$

$$\lambda^{-1} |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda |\xi|^2, \ \lambda = const > 0.$$

## References


1. Zhikov V., *Averaging of functionals of the calculus of variations and elasticity theory*, Izvestiya Akad. nauk SSSR. ser. mat. **50** (1986), no. 4, 4675 - 710.
2. Zhikov V., *On Lavrent'ev effect*, Dokl. Ross. Akad. nauk **345** (1995), no. 1, 675 - 710.
3. Zhikov V., *On Lavrentiev's Phenomenon*, Russian Journal of Math. Physics **3** (1995), no. 3, 249 - 269.
4. Zhikov V., *On Some Variational Problems*, Russian Journal of Math. Physics **5** (1996), no. 1, 105 - 116.
5. Xianling Fan., *A class of De Giorgi Type and Hölder Continuity of Minimizers of Variationals with* $m(x)$ *– Growth Condition*, Lanzhou University, China (1995).
6. Alkhutov Yu.A., *Harnack inequality and Hölder continuity of solutions of non - linear elliptic equations with non - standard growth condition*, Differentsial'nye Uravneniya (1997), no. 12 (to appear).
7. Trudinger N. S., *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721 - 747.
8. Acerbi E., Fusco N., *A transmission problem in the calculus of variations*, Calc. Var. **2** (1994), 1 - 16.
9. Kondratiev V.A., Landis E.M., *The qualitative theory of linear partial differential equations of the second order*, Sovremennie problemy matematiki. Fundamental'nie napravleniya, VINITI, **32** (1988), 99 - 215.
10. Wiener N., *Certain notions in potential theory*, J. Math. Phys. **3** (1924), 24 - 51.
11. Maz'ya V.G., *On the continuity at a boundary point of solutions of quasi - linear elliptic equations*, Vestnik Leningrad Univ. **3** (1976), 225 - 242.
12. Gariepy R., Ziemer W.P., *A regularity condition at the boundary for solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **67** (1977), 25 - 30.
13. Skrypnik I.V., *Methods of investigation of nonlinear elliptic boundary value problems*, Moscow, Nauka, 1990.
14. Kilpeläinen T., Malý J., *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137 - 161.

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