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Coarse rays

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Abstract. We give some characterizations of geodesic metric spaces coarsely equivalent to the ray $\mathbb{R}^+$.

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Let (X, d), (Y, ρ) be metric spaces. A mapping $f : X \to Y$ is called a *coarse embedding* if, for every $r > 0$, there exists $s > 0$ such that, for all $x_1, x_2 \in X$,

$$d(x_1, x_2) \leq r \Longrightarrow \rho(f(x_1), f(x_2)) \leq s,$$

$$\rho(f(x_1), f(x_2)) \leq r \Longrightarrow d(x_1, x_2) \leq s.$$

The metric spaces (X, d), (Y, ρ) are called *coarsely equivalent* if there exists a coarse embedding $f : X \to Y$ such that $f(X)$ is *large* in Y, i.e. there exists $t > 0$ such that, for every $y \in Y$, there is $z \in f(X)$ such that $\rho(y, z) \leq t$.

The space $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$ endowed with the Euclidian metric is called the *ray*. By a *coarse ray* we mean any metric space coarsely equivalent to ray.

For motivation of study metric spaces from the “coarse” point of view see [3, 5, 9, 10]. As is the segment $[0, 1]$ in topology, the ray is one of the simplest non-trivial objects in coarse geometry, so it is natural to ask for its characterization up to coarse equivalence. There is a simple test [5, Proposition 2.57] to recognize whether a given metric space is coarsely equivalent to some geodesic metric space. Thus, to answer this question we can work with only geodesic metric spaces. Theorem 1 and 3 provide several characterizations; Theorem 2 gives a better characterization in

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case of proper metric spaces; Example 2 shows that Theorem 2 does not hold for non-proper metric spaces. By Theorem 1, every geodesic metric space coarsely emenddable in the ray is a coarse ray. Clearly, it is not true outside of the geodesic case. Using the anti-Cantor set and Theorem 3.11 from [2], in Theorem 4 we describe the ultrametric spaces coarsely embeddable to the ray.

Let (X, d) be a metric space. We fix some point $x_0 \in X$ and define a preordering $\leq$ on X by the rule: $x \leq y$ if and only if $d(x_0, x) \leq d(x_0, y)$. For $\varepsilon \geq 0$, a space (X, d) is said to be ε-*directed* (with respect to the base point x_0) if, for any $x, y \in X$, $x \leq y$, we have

$$d(x_0, x) + d(x, y) \leq d(x_0, y) + \varepsilon.$$

If (X, d) is ε-directed then, for every $x' \in X$ there exists $\varepsilon' \geq 0$ such that (X, d) is ε'-directed with respect to x'.

Lemma 1. *Every ε-directed space is coarsely emenddable in the ray.*

Proof. Let (X, d) be ε-directed with respect to x_0. We define a mapping $f : X \to \mathbb{R}^+$ by the rule $f(x) = d(x_0, x)$, and note that, for any $x, y \in X$ with $x \leq y$, we have

$$d(x, y) - \varepsilon \leq f(y) - f(x) \leq d(x, y),$$

so f is a coarse embedding. □

By Theorem 1, the converse statement is true for every geodesic metric space (X, d), but in general case it does not hold.

Example 1. Let (X, d) be a half-parabola $\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : y = x^2\}$ with the metric d inherited from the plane. It is easy to see that the mapping $f : (X, d) \to \mathbb{R}^+$, $f(x, y) = y$ is a coarse embedding. On the other hand, (X, d) is not ε-directed for every $\varepsilon \geq 0$.

A subset Y of a metric space (X, d) is called *bounded* if there exists $C > 0$ such that *diam* $Y \leq C$ where *diam* $Y = \sup\{d(x, y) : x, y \in X\}$. A family $\Im$ of subsets of a metric space (X, d) is called *uniformly bounded* if there exists $C > 0$ such that *diam* $F \leq C$ for every $F \in \Im$.

Given a metric space (X, d) and any $x \in X$, $r \in \mathbb{R}^+$, we put

$$B(x, r) = \{y \in X : d(x, y) \leq r\}, S(x, r) = \{y \in X : d(x, y) = r\}$$

Lemma 2. *If* (X,d) *is an* ε*-directed space with the base point* x_0 *then the family* $\{S(x_0,r) : r \in \mathbb{R}^+\}$ *is uniformly bounded.*

Proof. Let $x, y \in X$, $d(x_0,x) = d(x_0,y)$ and $x \leq y$. Since $d(x_0,x) + d(x,y) \leq d(x_0,y) + \varepsilon$, we have $d(x,y) \leq \varepsilon$, so *diam* $S(x_0,r) \leq \varepsilon$ for every $r \in \mathbb{R}^+$. □

By Theorem 1 the converse statement is true for every geodesic metric space. On the other hand, Example 1 shows that in general case it does not hold.

Let (X,d), (Y,ρ) be metric spaces. Given $\lambda > 0$,$c \geq 0$, a mapping $f : X \to Y$ is called a (λ,c)*-isometric embedding* if, for all $x_1, x_2 \in X$,

$$\lambda^{-1}d(x_1,x_2) - c \leq \rho(f(x_1),f(x_2)) \leq \lambda d(x_1,x_2) + c.$$

If in addition $f(X)$ is large in Y, we say that f is a (λ,c)*-isometry.* The metric spaces (X,d), (Y,ρ) are called *quasi-isometric* if there exists a (λ,c)-isometry $f : X \to Y$.

Let (X,d) be a metric space, $r \geq 0$, $f : [0,r] \to X$ be an isometric embedding. We say that $f([0,r])$ is a *geodesic segment* with the endpoints $f(0), f(r)$. A metric space (X,d) is called *geodesic* if any two points of X can be joined by a geodesic segment.

Lemma 3. *If the geodesic metric spaces* $(X,d), (Y,\rho)$ *are coarsely equivalent then* $(X,d), (Y,\rho)$ *are quasi-isometric.*

Proof. Let $f : X \to Y$ be a coarse embedding such that $f(X)$ is large in Y. By [1, Proposition 1.4] or [5, Lemma 1.10], there exist λ, c such that $\rho(f(x),f(x')) \leq \lambda d(x,x') + c$ for all $x, x' \in X$.

Since $f(X)$ is large in Y, there exists $t > 0$ such that, for every $y \in Y$, we can find $y' \in f(X)$ with $\rho(y,y') < t$. We fix some points x, x' and some geodesic segment $[f(x), f(x')]$. On this segment we choose the points $y_1, \dots, y_n$ such that

$$y_1 = f(x), d(y_1,y_2) = \dots = d(y_{n-1},y_n) = 1, \rho(y_n, f(x')) = \varepsilon, \varepsilon < 1,$$

so $\rho(f(x),f(x')) = n+\varepsilon$. Then we pick the points $x_2, \dots, x_n \in X$ so that $\rho(f(x_2),y_2) < t, \dots, \rho(f(x_n),y_n) < t$. We put $s = 2t+1$, and choose $r > 0$ such that $\rho(f(a),f(b)) \leq s$ implies $d(a,b) \leq r$ for all $a, b \in X$. Then

$$\begin{aligned} d(x,x_1) &\leq d(x,x_2) + d(x_2,x_3) + \dots \\ &\qquad + d(x_{n-1},x_n) + d(x_n,x') \\ &\leq (n+1)r = (n+\varepsilon)r + r(1-\varepsilon) \\ &\qquad = r\rho(f(x),f(x')) + r(1-\varepsilon). \end{aligned}$$

Since the choice of r does not depend on $x, x^{'}$, in view of the first paragraph, we conclude that f is a quasi-isometry. □

Theorem 1. *For an unbounded geodesic metric space* (X, d), $x_0 \in X$, *the following statements are equivalent:*

- (i) (X, d) *is a coarse ray;*
- (ii) (X, d) *is coarsely emenddable in the ray;*
- (iii) (X, d) *is* ε*-directed;*
- (iv) *the family* $\{S(x_0, r) : r \in \mathbb{R}^+\}$ *is uniformly bounded.*

Proof. $(i) \Leftrightarrow (ii)$. The implication $(i) \Rightarrow (ii)$ is trivial. To check $(ii) \Rightarrow (i)$, we fix some coarse embedding $f : X \to \mathbb{R}^+$. Then we pick $\lambda > 0$ such that $d(a, b) \le 1$ implies $|f(a) - f(b)| < \lambda$. Given an arbitrary points $x, x^{'} \in X$, we choose the points $x_1, \ldots, x_n$ on the geodesic segment $[x, x^{'}]$ such that $x = x_1$, $d(x_1, x_2) = \ldots d(x_{n-1}, x_n) = 1$, $d(x_n, x^{'}) < 1$. Then every segment of length λ on $[f(x), f(x^{'})]$ contains at least one point $f(x_1), \ldots, f(x_n)$. Since $f(X)$ is unbounded in $\mathbb{R}^+$, it follows that $f(X)$ is large, so we get (i).

$(i) \Rightarrow (iii)$ Let f be a coarse embedding of (X, d) into $\mathbb{R}^+$ such that $f(X)$ is large in $\mathbb{R}^+$. By Lemma 3, f is a (λ, C)-isometric embedding. Changing the value of f in x_0 we get a (λ', C')-isometric embedding for some parameters λ', C', so we may suppose that $f(x_0) = 0$ and $f(x_0) \le f(x)$ for every $x \in X$. We put $g = \frac{1}{\lambda} f$, $C_1 = \frac{C}{\lambda}$. Then g is $(1, C_1)$ isometric embedding of (X, d) into $\mathbb{R}^+$ and, for all $x, y \in X$, we have

$$d(x, y) - C_1 \le |g(x) - g(y)| \le d(x, y) + C_1$$

$$|g(x_0) - g(x)| - C_1 \le d(x_0, x) \le |g(x_0) - g(y)| + C_1.$$

Now let $d(x_0, x) \le d(x_0, y)$. Then

$$g(x) - g(x_0) - C_1 \le g(y) - g(x_0) + C_1, g(x) - g(y) \le 2C_1.$$

If $g(y) \ge g(x)$ we have

$$\begin{aligned} d(x_0, x) + d(x, y) &\le |g(x_0) - g(x)| + |g(x) - g(y)| + 2C_1 \\ &= g(x) - g(x_0) + g(y) - g(x) + 2C_1 \\ &= |g(y) - g(x_0)| + 2C_1 \le d(x_0, y) + 3C_1. \end{aligned}$$

If $g(y) \leq g(x)$ then $g(x) - g(y) \leq 2C_1$, and we have

$$d(x_0, x) + d(x, y) \leq |g(x_0) - g(x)| + |g(x) - g(y)| + 2C_1$$
$$\leq |g(x_0) - g(x)| + 4C_1 \leq d(x_0, x) + 5C_1 \leq d(x_0, y) + 5C_1.$$

In both cases we see that (X, d) is a $5C_1$-ray.

(iii) $\Rightarrow$ *(iv)* follows from Lemma 2.

(iv) $\Rightarrow$ *(iii)*. We choose $\varepsilon > 0$ such that *diam* $S(x_0, r) \leq \varepsilon$ for every $r \geq 0$. Let $x, y \in X$ and $d(x_0, x) \leq d(x_0, y)$. Since (X, d) is geodesic, there exists a point x' on the geodesic segment $[x_0, y]$ such that $d(x_0, x) = d(x_0, x')$. Then

$$d(x_0, x) + d(x, y) \leq d(x_0, x') + d(x, x') + d(x', y)$$
$$= d(x, y) + d(x, x') \leq d(x_0, y) + \varepsilon.$$

(iii) $\Rightarrow$ *(ii)* follows from Lemma 1. □

A subspace L of a metric space is called a *geodesic ray* if L is an isometric copy of $\mathbb{R}^+$.

An unbounded metric space (X, d) is called *proper* if every closed ball $B(x, r)$ in (X, d) is compact.

The next lemma is a geodesic version of Kønig Lemma stating that every infinite locally finite graph has an infinite chain.

Lemma 4. *Every proper geodesic metric space (X, d) has a geodesic ray.*

Proof. We use the Hausdorff distance d_H defined on the set $\mathcal{C}(X)$ of all compact subsets of (X, d) by the rule

$$d_H(C, C') = \inf\{\varepsilon > 0 : C \subseteq B(C', \varepsilon), C' \subseteq B(C, \varepsilon)\},$$

where $B(C, \varepsilon) = \bigcup_{c \in C} B(c, \varepsilon)$. By [5, Proposition 7.2], $\mathcal{C}(Y)$ is compact for every compact metric space Y. We fix an arbitrary point $x_0 \in X$ and, for every $n \in \omega$, pick $x_n \in X$ such that $d(x_0, x_n) = n$. For every $n \in \omega$, we choose a geodesic segment $[x_0, x_n]$. Since every space $\mathcal{C}(B(x_0, m)), m \in \omega$ is compact, there exists a subsequence $(n_k)_{k \in \omega}$ of ω such, that for every $m \in \omega$ the sequence $(B(x_0, m) \cap [x_0, x_{n_k}])_{k \in \omega}$ converges to some subset L_m. It is a standard verification that $L = \bigcup_{m \in \omega} L_m$ is a geodesic ray. □

Theorem 2. *A proper geodesic metric space (X, d) is a coarse ray if and only if (X, d) has a large geodesic ray.*

Proof. Let (X, d) be a coarse ray. By Lemma 4, (X, d) has a geodesic L. By the equivalence $(i) \Leftrightarrow (iv)$ of Theorem 1, L is large in (X, d). On the other hand, if (X, d) has a large geodesic ray then (X, d) is a coarse ray by definition. □

Example 2. We construct a geodesic (non-proper) coarse ray (X, d) which has no geodesic rays. To this end we take a disjoint family $\{[a_n, b_n] : n \in \omega\}$ of segments of length n, and stick together all the points $a_n, n \in \omega$. Denote by a the resulting point. Then, for every $m \in \omega$, we choose the points $x_n, n \in \omega, n \geq m$ such that $x_n \in [a, b_n]$, $|[a, x_n]| = m$, and connect any two points x_n, x_k, $m \leq n < k$ by the segment of length 1. Let X be the resulting set endowed with the path metric d. By the equivalence $(i) \Leftrightarrow (iv)$ of Theorem 1, (X, d) is a coarse ray. To see that (X, d) has no geodesic rays it suffices to observe that every geodesic segment connecting the points a, x, where $x \in [a, b_n]$, lies on the segment $[a, b_n]$.

We say that a subspace L of a metric space (X, d) is an *almost geodesic ray* if there exist $c \geq 0$ and a bijection $f : \mathbb{R}^+ \to L$ such that, for all $t_1, t_2 \in \mathbb{R}^+$, we have

$$|t_2 - t_1| \leq d(f(t_1), f(t_2)) \leq |t_2 - t_1| + c.$$

Clearly, every almost geodesic ray is a coarse ray.

Theorem 3. *A geodesic metric space (X, d) is a coarse ray if and only if (X, d) has a large almost geodesic ray.*

Proof. Let (X, d) be a coarse ray, $x_0 \in X$. By Theorem 1, there exists $C \geq 0$ such that $\operatorname{diam} S(x_0, t) \leq C$ for every $t \in \mathbb{R}^+$. Since (X, d) is geodesic, for every $t \geq 0$, the set $S(x_0, t)$ is non-empty, so we can take some point $f(t) \in S(x_0, t)$ and get the mapping $f : \mathbb{R}^+ \to X$. We put $L = f(\mathbb{R}^+)$, note that L is large in (X, d) and show that L is an almost geodesic ray. Let $t_1, t_2 \in \mathbb{R}^+$ and $t_1 \leq t_2$. Since (X, d) is geodesic, there exists $y \in S(x_0, t_1)$ such that $d(f(t_2), z) = t_2 - t_1$. Then

$$t_2 - t_1 \leq d(f(t_1), f(t_2)) \leq d(f(t_1), z) + d(z, f(t_2)) \leq (t_2 - t_1) + C.$$

On the other hand, if L is a large almost geodesic ray in (X,d), then L is a coarse ray, so (X,d) is also a coarse ray. □

For $r>0$, a subset Y of a metric space (X,d) is called *r-discrete* if $d(a,b)\geq r$ for any $a,b\in Y,\ a\neq b$. The *r-capacity* of Y is the cardinal $\sup\{|Z| : Z$ is r-discrete subset of Y $\}$. A metric space (X,d) is of *bounded geometry* if there exists a number $r>0$ and a function $c:\mathbb{R}^+\to\mathbb{R}^+$ such that the r-capacity of every ball $B(x,t)$ does not exceed $c(t)$.

A metric d on a set X is called *ultrametric* if, for all x,y,z,

$$d(x,y)\leq \max\{d(x,y),d(z,y)\}$$

Theorem 4. *An ultrametric space (X,d) is coarsely emenddable in the ray if and only if (X,d) is of bounded geometry.*

Proof. Let (X,d) be a space of bounded geometry. We fix the corresponding $r>0$, $c:\mathbb{R}^+\to\mathbb{R}^+$ and choose a maximal r-discrete subspace Y of X. Given any $x\in X$, there exists $y\in Y$ such that $d(x,y)\leq r$. It follows that Y is large in (X,d), so Y is coarsely equivalent to (X,d). Every ball of radius t in Y has at most $c(t)$ points, in particular, Y is a proper metric space. By [2, Theorem 3.11], Y is coarsely emenddable into the subspace M of $\mathbb{R}^+$ consisting of all integers whose tercimal decomposition does not contain 1. Since X is coarsely equivalent to Y, there exists a coarse embedding of X into $\mathbb{R}^+$.

On the other hand, let (X,d) be coarsely equivalent to some subspace Z of $\mathbb{R}^+$. Since Z is of bounded geometry, it is easy to check that (X,d) is also of bounded geometry. □

Problem. *Detect all metric spaces coarsely emenddable in the ray $\mathbb{R}^+$.*

References

[1] A. Dranishnikov, *Asymptotic topology* // Russian Math. Surveys, **55** (2000), N 6, 71–116.

[2] A. Dranishnikov, M. Zarichnyi, *Universal spaces for asymptotic dimension* // Topol. Appl. **140** (2004), N 2–3, 203–225.

[3] M. Gromov, *Asymptotic invariants of infinite groups*, London Math. Soc. Lecture Note Ser. **182**, 1993.

[4] I. Protasov, M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Sec., **13**, 2006.

[5] J. Roe, *Lectures on Coarse Geometry*, AMS University Lecture Series, **31**, 2003.

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