THE APPLICATION OF THE INTERSECT INDEX TO QUASILINEAR EIGENFUNCTION PROBLEMS

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First time the intersect index was applied to non-linear problems in L. Lusternick's research. This direction is investigating at voronezh school now [1,2]. Small eigenfunctions and its global branches was considered by the intersect index in [3-5].

1. Definitions. We are interested in eigenvalues (e.v.) $\lambda \in \mathbf{R}$ and eigenfunctions (e.f.) $u \in W_2^1(\Omega)$ of the quasilinear problem

$$\Delta u + p(u, grad(u), x)u + \lambda u = 0, \qquad u|_{\partial\Omega} = 0 \tag{1}$$

$$u \in S_R^\infty = \{u : \int_\Omega u^2 = R^2\} \qquad (R > 0), \tag{2}$$

where $W_2^k(\Omega)$ is Sobolev's space with norm $\|\cdot\|_k$, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $x \in \overline{\Omega}$, Δ is Laplas operator, p is a continue function. For simplicity of a priori estimates we have to suppose $m < p(u, y, x) < M \quad ((u, y, x) \in \mathbf{R}^{n+1} \times \overline{\Omega})$.

The pair (λ, u) which satisfy (1),(2) is called *normalised solution* (n.s.). If (λ^*, u^*) is a n.s. then λ^* is an e.v. of the linear problem

$$\Delta u + q(x)u + \lambda u = 0, \quad u|_{\partial\Omega} = 0, \tag{3}$$

where

$$q(x) = p(u^*(x), grad(u^*(x)), x). \tag{4}$$

The e.f. u^* is among eigenfunctions of the problem (3),(4) certainly. The linear problem (3),(4) is symmetric that is why $\lambda \in \mathbf{R}$. Eigenvalues of (3) form the nondecreasing sequence $\lambda_0 < \lambda_1 \leq \lambda_2 \leq ...;\ \lambda_n \to \infty$.

D e f. 1. The n.s. (λ^*, u^*) of the problem (1),(2) is named *the simple (n-multiple)* , if λ^* is simple (n-multiple) for the linear problem (3),(4). (The multiplicity of e.v. is finite always.)

D e f. 2. The n.s. (λ^*, u^*) of the problem (1),(2) and its elements have such number which the e.v. λ^* has as a eigenvalue of the linear problem (3),(4).

We need a priori estimates of normalised solutions which have bounded numbers.

L e m m a 1. *Eigenvalues λ with number n of the problem (1),(2) satisfy estimates $\lambda_n - M < \lambda < \lambda_n + m$ where λ_n is the e.v. with number n of the problem (3) with $q(x) \equiv 0$.*

L e m m a 2. *Normalised solutions (λ, u) with number n of the problem (1),(2) satisfy the estimate $|\lambda| + \|u\|_2 < C$ where the constant C depends on R, n, m, M only.*

The problem (1) is equal to the operator equation

$$u + (\lambda + M)A(u)u = 0, \tag{5}$$

due to lemma 1 where A is a continue mapping from $W_2^1(\Omega)$ to Banach space L of linear symmetric compact operators.

We consider the family of linear equations

$$u + (\lambda + M)Bu = 0. \tag{6}$$

An operator $B \in L$ is the parameter of the family. Let $T_R^\infty = \{(B,u) \in L \times S_R^\infty :$ *u is an e.f. of the problem* $(6)\}$. The set T_R^∞ is a smooth Banach manifold with model space L [6]. The manifold T_R^∞ is stratificated by numbers and multiplicity of its eigenfunctions: $T_R^\infty(n,l) = \{(B,u) \in T_R^\infty : u$ *is an e.f. of (6) with e.v.* λ, *moreover* $\lambda_{n-1}(B) < \lambda = \lambda_n(B) = ... = \lambda_{n+l-1}(B) < \lambda_{n+l}(B)\}$. Thus $T_R^\infty = \bigcup_{n,l\in N} T_R^\infty(n,l)$. According to [7] it's possible to prove that $T_R^\infty(n,l)$ is the smooth submanifold of T_R^∞ end $codim T_R^\infty(n,l) = (l-1)l/2$. Notice $codim T_R^\infty(n,1) = 0,\ codim T_R^\infty(n,2) = 1$. We give those number end multiplicity to a point $(B,u) \in T_R^\infty$ which the e.f. u has.

We examine the mapping

$$Gr_A:\ S_R^\infty \longrightarrow L \times S_R^\infty, \qquad Gr_A(u) = (A(u),u), \tag{7}$$

which is important for us.

T h e o r e m 1. *A function u is an e.f. of the equation (6) only in the case $Gr_A(u) \in T_R^\infty$. The number of solution (λ,u) and its multiplicity are defined by the index (n,l) of stratum $T_R^\infty(n,l)$: $Gr_A(u) = (A(u),u) \in T_R^\infty(n,l) \subset T_R^\infty$.*

D e f. 3. A mapping A is called *n-typical* if the image of the mapping (7) doesn't intersect stratums $T_R^\infty(n,l)$ where the multiplicity $l \geq 2$. Other words solutions with number n are simple.

We will show that simple solutions can be obtained by the intersect index.

2. Intersect index. At first we consider the finite dimensional problem

$$v + \gamma K(v)v = 0, \quad v \in S^{k-1}, \tag{8}$$

which is analogous to the problem (5); K is a continue mapping from S^{k-1} to the space L^k of real symmetric k-dimensional matrixes. Definitions 1-3 have the sense in the problem (8). Manifolds $T^k,\ T^k(n,l)$, the mapping Gr_K are determined by analogy with $T_R^\infty,\ T_R^\infty(n,l),\ Gr_A$ accordingly. Theorem 1 is true in case of the problem (8).

L e m m a 3. *The set of n-typical mappings K is opened and dense in the space of continue mappings from S^{k-1} to L^k.*

Since $dim T^k = dim L^k$ for any $n \leq k$ and an n-typical mapping K is determined the orientated intersect index $\chi(\overline{T}^k(n,1),\ Gr_K) = \chi(n,K)$ $(\overline{T}^k(n,1)$ is the closure of the stratum $T^k(n,1))$. If the index isn't equal to zero then the equation (8) has a n.s. with number n. The calculation of the index is a difficult problem due to the manifold $\overline{T}^k(n,1)$ has the boundary.

Let $\{u_0, u_1, ...\}$ be the set of eigenfunctions of some operator $B \in L$. Let $\mathbf{R}^k \subset W_2^1(\Omega)$ $(k = 1, 2...)$ be the finite dimensional subspace which is generated by the basis

$\{u_0, u_1, ..., u_{k-1}\}$. Let P^k be the orthogonal projection on $\mathbf{R}^k$. We replace the problem (5),(2) by the approximate equation

$$v + (\lambda + M)P^k A(v)v = 0, \quad v \in S^{k-1}, \tag{9}$$

which has type of (8). If a mapping A is n-typical than the mapping P^kA is n-typical for any big k too. Therefore the index $\chi(n, P^kA)$ is determined for any big k.

T h e o r e m 2. *Index $\chi(n, P^kA)$ has not change for any big k.*

D e f. 4. Let L be a n-typical mapping. We determine that the orientated intersect index $\chi(\overline{T}_R^\infty(n, 1),\ Gr_A) = \chi(n, P^kA)$, where k is big enough.

If the index isn't equal to zero then the problem (5),(2) has a n.s. with number n. Moreover, the solution is the limit ($k \to \infty$) of solutions of equations (9) due to a priory estimates (lemma 2).

The intersect index is an invariant of a homotopy in the class of n-typical mappings. In our opinion a control of n-typeness isn't easy. For small eigenfunctions n-typeness are checked in a finite dimensional kernel of the linear problem

$$\Delta u + p(0, 0, x)u + \lambda^* u = 0, \qquad u|_{\partial\Omega} = 0, \tag{10}$$

where λ^* is the e.v. of the problem (10) [3,4].

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