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REGULAITY OF INFINITE DIMENSIONAL HEAT DYNAMICS OF UNBOUNDED LATTICE SPINS WITH NON-CONSTANT DIFFUSION COEFFICIENTS

Below we demonstrate how the C^∞-regular properties of heat dynamics with non-unit nonlinear diffusion coefficient can be studied. We consider an infinite dimensional model, describing evolution of unbounded lattice spins $\mathbb{R}^{\mathbb{Z}^d}$. As a main step we provide a construction of corresponding variational processes in $\ell_p(c)$ spaces with growing weights $c_k \sim e^{a|k|},\ k \in \mathbb{Z}^d$.

Developing the approach of nonlinear estimates on variations, we find sufficient conditions on the nonlinear coefficients of differential equation that lead to C^∞-regularity of solutions with respect to the initial data and C^∞-regularity of corresponding heat semigroup.

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1. Introduction.

It is already known, e.g. [7, 8], that for the stochastic differential equations

$$dy^0 = B(y^0)dW_t - F(y^0)dt, \quad y^0(0) = x^0 \tag{1}$$

with coefficients, that are globally Lipschitz and have all bounded derivatives, there is C^∞-regularity of solutions $y^0_t(x^0)$ with respect to the initial data x^0. Moreover, corresponding heat semigroup, defined as a mean $P_t f(x^0) = \mathbf{E}\, f(y^0_t(x^0))$ with respect to the Wiener measure, preserves spaces of continuously differentiable functions with bounded derivatives. These results follow from application of fixed point and implicit function theorems to variations $y^j_t(x) = \dfrac{\partial^j y^0_t(x^0)}{\partial (x^0)^j}$ of process $y^0_t(x^0)$ with respect to the initial data x^0.

The consideration of more wide class of stochastic differential equations with essentially nonlinear non-Lipschitz coefficients leads

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to a monotone conditions of coercitivity and dissipativity: $\forall C > 0\ \exists M$ such that

$$\textbf{coercitivity} \quad < F(x) - F(y), x - y > - \\ -C\|B(x) - B(y)\|^2 \geq -M\|x - y\|^2$$

$$\textbf{dissipativity} \quad < F(x), x > -C\|B(x)\|^2 \geq -M(1 + \|x\|^2),$$

that are sufficient for the existence, uniqueness and continuous dependence of solutions with respect to the initial data [10, 11].

In [2, 4, 5] it was shown that the application of Cauchy-Liouville-Picard scheme to the problem of C^∞-regularity for non-Lipschitz differential equations meets difficulties. Here we discussed a particular case of system (1) with constant diffusion coefficient $B = 1$, that has important applications to the classical Gibbs lattice systems with unbounded spins. To be able to work with such nonlinear differential equations we followed [8, 9], where, after the shift $\eta_t = y_t - W_t$, equation (1) becomes ordinary differential equation on variable η_t:

$$d\eta_t = -F(\eta_t + W_t)dt$$

with random control W_t.

In [2, 4, 5] we found that due to the structure of the associated with (1) variational system

$$\begin{cases} dy^i = \sum\limits_{\substack{j_1+\ldots+j_s=i,\\ s\geq 1}} B^{(s)}(y^0)y^{j_1}..y^{j_s}dW - \sum\limits_{\substack{j_1+\ldots+j_s=i,\\ s\geq 1}} F^{(s)}(y^0)y^{j_1}..y^{j_s}dt \\ y^1(0) = Id, \quad y^i(0) = 0,\ i \geq 2 \end{cases} \tag{2}$$

the variation of N^{th} order is proportional to the N^{th} power of the variation of 1^{st} order.

Such proportionality led to nonlinear estimates on variations

$$\rho_n(t) = \sum_{j=1}^{n} \mathbf{E}\, p_j(\|y(t)\|)\, \|y^j(t)\|_{X_j}^{m/j} \leq e^{Mt}\rho_n(0), \tag{3}$$

permitting to apply monotone methods to the problem of C^∞-regularity. The weights p_j and topologies X_j on variations were found to be related with the order of nonlinearity of coefficients of initial equation

(1). Moreover, the order of nonlinearity also influenced the structure of topologies in the spaces of differentiable functions, preserved by heat semigroup P_t.

In [2] it was observed that the variations should be constructed in spaces $\ell_p(c)$ with exponentially growing on lattice $\mathbb{Z}^d$ weights, i.e. $c_k \sim e^{a|k|}$, $k \in \mathbb{Z}^d$. For diffusion coefficient $B = I$ this property follows from Kato results about the construction of solutions to the linear ordinary differential equations. For $B = I$ terms with $B^{(s)} = 0$ for $s \geq 1$ in (2) are absent and (2) becomes non-autonomous inhomogeneous linear equation on variable y^i with control y^0_t.

The use of process η and application of Kato results becomes impossible for non-constant diffusion coefficient $B \neq I$. The solution of this problem is a main topic of this article.

In Section 2 we describe a model with non-constant nonlinear diffusion coefficient and state main results about the properties of variations of diffusion process and regularity of its semigroup. In Section 3 we define the stochastic integrals $\int_0^t B_s dW_s$ with $B \in \ell_p(c)$ and construct the nonlinear diffusion and its variations with respect to the initial data. In Section 4 we prove nonlinear estimate (3). Section 5 is devoted to the study of continuity and C^∞ regularity of variations with respect to the initial data. Here we also demonstrate the regularity of heat semigroup P_t (proof of Theorem 1).

Finally remark, that even the problem of the first order regularity with respect to the initial data is still under question for more general classes of stochastic differential equations, e.g. [6] and references therein.

2. Basic model and statement of main results.

We consider the stochastic process on the lattice product of spin spaces $\mathbb{R}^{\mathbb{Z}^d} = \prod_{k=(k_1,\dots,k_d)\in\mathbb{Z}^d} \mathbb{R}^1$, described by the following nonlinear equation

$$y^0(t) = x^0 + \int_0^t B(y^0(s))dW(s) - \int_0^t [F(y^0(s)) + Ay^0(s)]ds \qquad (4)$$

Nonlinear diagonal maps

$$\mathbb{R}^{\mathbb{Z}^d} \ni x = \{x_k\}_{k\in\mathbb{Z}^d} \longrightarrow B(x) = \{B(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$$

$$\mathbb{R}^{\mathbb{Z}^d} \ni x = \{x_k\}_{k\in\mathbb{Z}^d} \longrightarrow F(x) = \{F(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$$

are generated by smooth functions $B, F \in C^\infty(\mathbb{R}^1)$ of polynomial with derivatives behaviour and the linear finite diagonal map $A : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is defined by

$$\exists r_0 \qquad (Ax)_k = \sum_{j:\,|j-k|\le r_0} A(k-j)x_j, \qquad k \in \mathbb{Z}^d$$

and is bounded in any space $\ell_p(c)$, $\sup_{|k-j|=1} |c_k/c_j| < \infty$.

The cylinder Wiener process $W = \{W_k(t)\}_{k\in\mathbb{Z}^d}$ with values in $\ell_2(a)$, $\sum_{k\in\mathbb{Z}^d} a_k = 1$, $a \in \mathbb{P}$ is canonically realized on measurable space $(\Omega = C_0([0,T], \ell_2(a)), \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with canonical filtration $\mathcal{F}_t = \sigma\{W(s) | 0 \le s \le t\}$ and cylinder Wiener measure $\mathbf{P}$. Processes W_k, $k \in \mathbb{Z}^d$ are independent $\mathbb{R}^1$-valued Wiener processes. Henceforth we denote by $\mathbf{E}$ the expectation with respect to measure $\mathbf{P}$ and by $\mathbb{P}$ the set of all vectors $a = \{a_k\}_{k\in\mathbb{Z}^d}$ such that $\delta_a = \sup\limits_{|k-j|=1} |a_k/a_j| < \infty$.

Let us impose the following conditions on the coefficients $\{F, B\}$.

1. **Coercitivity and dissipativity:** $\forall M \; \exists K_M,\ K_1,\ K_2$ such that

$$(x-y)(F(x)-F(y)) - M(B(x)-B(y))^2 \ge K_M(x-y)^2 \quad (5)$$

$$xF(x) - M\,B^2(x) \ge -K_1x^2 - K_2 \quad (6)$$

Inequality (5) implies in particular that $\forall M \; \exists K_M$

$$-F'(x) + M[B'(x)]^2 \le K_M \quad (7)$$

2. **Nonlinear parameters:** Function $F : \mathbb{R}^1 \to \mathbb{R}^1$ is monotone and $\exists \mathbf{k}_F, \mathbf{k}_B \ge -1$ with $2\mathbf{k}_B \le \mathbf{k}_F$ such that $\forall n \in \mathbb{N} \; \exists C_n$ $\forall i = 0, ..., n \; \forall x, y \in \mathbb{R}^1$

$$|F^{(i)}(x) - F^{(i)}(y)| \le C_n|x-y|(1+|x|+|y|)^{\mathbf{k}_F} \quad (8)$$

$$|B^{(i)}(x) - B^{(i)}(y)| \le C_n|x-y|(1+|x|+|y|)^{\mathbf{k}_B} \quad (9)$$

Main result is that under the above conditions the heat diffusion semigroup

$$(P_tf)(x) = \mathbf{E}\, f(y^0(t,x^0)) \quad (10)$$

preserves spaces of continuously differentiable functions, which topologies depend on the order of nonlinearity $\mathbf{k}_F$. This result generalizes [1, 3, 2], where the unit diffusion case $B(x) = 1$ was considered.

Let us say that array $\Theta = \Theta^1 \cup ... \cup \Theta^n,\ n \in I\!N$ with Θ^m be a set of pairs of m^{th}-order $(p, \mathcal{G} = G^1 \otimes ... \otimes G^m),\ G^i \in I\!P,\ i = 1, ..., m$, is *quasi-contractive with parameter* $\mathbf{k}_F$ if $\forall m = 2, ..., n\ \forall (p, \mathcal{G}) \in \Theta^m$ and $\forall i, j \in \{2, ..., m\},\ i < j$ there is a pair $(\widetilde{p}, \widetilde{\mathcal{G}} = \widetilde{G}^1 \otimes ... \otimes \widetilde{G}^{m-1}) \in \Theta^{m-1}$ such that $\exists K \in I\!R_+$

$$\forall z \in I\!R_+ \quad (1+z)^{\frac{\mathbf{k}_{F+1}}{2}} \widetilde{p}(z) \le K\, p(z) \tag{11}$$

$$(\widehat{\mathcal{G}}^{\{i,j\}})^\ell \le K\, \widetilde{G}^\ell, \quad \ell = 1, ..., m-1 \tag{12}$$

Above $p, \widetilde{p}$ are smooth functions of polynomial behaviour (27) and inequality (12) is understood as a coordinate inequality between $(m-1)^{th}$ order tensors for $(m-1)$-tensor

$$\widehat{\mathcal{G}}^{\{i,j\}} = G^1 \otimes ... \otimes G^{i-1} \otimes G^{i+1} \otimes ... \otimes G^{j-1} \otimes a^{-(\mathbf{k}_F+1)} G^i G^j \otimes G^{j+1} \otimes ... \otimes G^m$$

constructed by m-tensor $\mathcal{G} = G^1 \otimes ... \otimes G^m$.

Definition 1. *Function* $f \in \mathcal{D}_{\Theta,r}(\ell_2(a)),\ r \ge 0$, *iff*
1. *There is a set of Borel measurable partial derivatives*

$$\ell_2(a) \ni x \to \partial_\tau f(x) \in I\!R^1 \qquad \forall \tau = \{j_1, ..., j_s\}, \quad |\tau| \le n \tag{13}$$

such that $\forall x^0 \in \ell_2(a),\ \forall h \in \mathbf{AC}([a, b])$

$$f(x^0 + h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k \in Z\!\!Z^d} \partial_k f(x^0 + h(s)) h_k'(s) \tag{14}$$

and $\forall \tau \quad |\tau| \le n-1$

$$\partial_\tau f(x^0 + h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k \in Z\!\!Z^d} \partial_{\tau \cup \{k\}} f(x^0 + h(s)) h_k'(s) \tag{15}$$

Here we used notation

$$\mathbf{AC}([a, b]) = \bigcap_{p \ge 1,\ c \in I\!P} AC([a, b], \ell_p(c)) \tag{16}$$

for

$$AC([a,b],X) = \{h \in C([a,b],X) \ : \ \exists h' \in L_1([a,b],X)\}$$

2. *The norm is finite*

$$\|f\|_{\mathcal{D}_{\Theta,r}} = \|f\|_{Lip_r} + \max_{m=1,...,n} \|\partial^{(m)} f\|_{\Theta^m} < \infty \qquad (17)$$

where

$$\|f\|_{Lip_r} = \sup_{x\in\ell_2(a)} \frac{|f(x)|}{(1+\|x\|_{\ell_2(a)})^{r+1}} + \qquad (18)$$

$$+ \sup_{x,y\in\ell_2(a)} \frac{|f(x)-f(y)|}{\|x-y\|_{\ell_2(a)}(1+\|x\|_{\ell_2(a)}+\|y\|_{\ell_2(a)})^r}$$

and for multifunction of m^{th} *order* $\partial^{(m)} f(x) = \{\partial_\tau f(x), \ |\tau| = m\}$

$$\|\partial^{(m)} f\|_{\Theta^m} = \sup_{x\in\ell_2(a)} \max_{(p,\mathcal{G})\in\Theta^m} \frac{|\partial^{(m)} f(x)|_{\mathcal{G}}}{p(1+\|x\|^2_{\ell_2(a)})} \qquad (19)$$

with $|\partial^{(m)} f(x)|^2_{\mathcal{G}} = \sum_{\tau=\{j_1,...,j_m\}\subset\mathbb{Z}^d} G^1_{j_1}...G^m_{j_m} |\partial_\tau f(x)|^2$ *for* $\mathcal{G} = G^1 \otimes ... \otimes G^m$.

Theorem 1. *Let* F, B *satisfy conditions (5)-(9) and* $\Theta = \Theta^1 \cup ... \cup \Theta^n$, $n \in \mathbb{N}$ *be quasi-contractive array with parameter* $\mathbf{k}_F$. *Suppose that function* $f \in \mathcal{D}_{\Theta,r}(\ell_2(a))$, $r \geq 0$, *i.e.*
Then $\forall \geq> 0$ *semigroup* P_t *preserves scale of spaces* $\mathcal{D}_{\Theta,r}(\ell_2(a))$, $r > 0$ *and there are* $K_{\Theta,r}$, $M_{\Theta,r}$ *such that*

$$\forall f \in \mathcal{D}_{\Theta,r}(\ell_2(a)) \qquad \|P_t f\|_{\mathcal{D}_{\Theta,r}} \leq K_{\Theta,r} e^{M_{\Theta,r} t} \|f\|_{\mathcal{D}_{\Theta,r}} \qquad (20)$$

The formal differentiation of (10) with respect to x^0 shows that the derivatives of semigroup is related with the variations of process y^0_t with respect to the initial data x^0. Let $\tau = \{j_1,...,j_n\}$, $j_s \in \mathbb{Z}^d$ be any ordered array of points from $\mathbb{Z}^d$. To the set τ we associate vector $y_\tau = \{y_{k,\tau}\}_{k\in\mathbb{Z}^d}$, which satisfies equation

$$\begin{aligned} y_{k,\tau} = \widetilde{x}_{k,\tau} + \textstyle\int_0^t (B'(y^0_k) y_{k,\tau} + \varphi^B_{k,\tau}) dW_k - \\ \textstyle\int_0^t (F'(y^0_k) y_{k,\tau} + (Ay_\tau)_k + \varphi^F_{k,\tau}) ds, \quad k \in \mathbb{Z}^d, \end{aligned} \qquad (21)$$

derived by differentiation of (4) with respect to variables $\{x^0_{j_n}, ..., x^0_{j_1}\}$. Above the inhomogeneous parts φ^B_τ and φ^F_τ are constructed from functions B and F by the following rule

$$\varphi^D_{k,\tau} = \sum_{\gamma_1\cup...\cup\gamma_s=\tau,\ s\geq 2} D^{(s)}(y^0_k)y_{k,\gamma_1}...y_{k,\gamma_s}, \tag{22}$$

where $y_{\gamma_1}, ..., y_{\gamma_s}$ are the solutions of lower rank variational equations. Summation in (22) runs on all possible subdivisions of set $\tau=\{j_1,..,j_n\}$ on the nonintersecting subsets $\gamma_1, ..., \gamma_s \subset \tau,\ |\gamma_1|+...+|\gamma_s| = |\tau|,\ s \geq 2,\ |\gamma_i| \geq 1$.

To prove Theorem 1 it is necessary to find the joint topologies for solvability of system in variations (21), and to check that at the special choice of initial data in (21)

$$\widetilde{x}_{k,\tau} = \delta_{kj} \quad \text{for} \quad \tau = \{j\},\ |\tau| = 1 \quad \text{and} \quad \widetilde{x}_{k,\tau} = 0 \quad \text{for} \quad |\tau| \geq 2 \tag{23}$$

the variation y_τ is interpreted as a derivative of y^0 with respect to x^0

$$\frac{\partial^{|\tau|} y^0_k(t,x^0)}{\partial x^0_{j_n}...\partial x^0_{j_1}} = y_{k,\tau} \tag{24}$$

Equation (21) possesses a certain nonlinear symmetry with respect to the lower rank variations, where the i^{th} order variation and the i^{th} degree of the first order variation appear simultaneously. Like in [2] introduce the following nonlinear object

$$\rho_\tau(y;t) = \mathbf{E}\sum_{i=1}^{n} p_i(z_t) \sum_{\gamma\subset\tau,\ |\gamma|=i} \|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \tag{25}$$

where the set $\tau = \{j_1, ..., j_n\},\ j_i \in \mathbb{Z}^d,\ z_t = 1 + \|y^0(t,x^0)\|^2_{\ell_2(a)}$ and $m_\gamma = m_1/|\gamma|$.

Impose the following hierarchy of weights p_i, c_τ. It is dictated by the unbounded operator coefficients with control y^0 in (21), (22) and depends on the order of nonlinearity $\mathbf{k}_F \geq 2\mathbf{k}_B$:

1. The vectors $c_\gamma = \{c_{k,\gamma}\}_{k\in\mathbb{Z}^d} \subset \mathbb{P}$ fulfill

$$\forall\alpha\subset\tau\ \forall\gamma_1\cup...\cup\gamma_s=\alpha\ \forall s\geq 2\ \exists K_{\gamma_1,...,\gamma_s;\alpha} \ \ \textit{such that}\ \forall k\in\mathbb{Z}^d$$

$$[c_{k,\alpha}]^{|\alpha|} a_k^{-\frac{\mathbf{k}_{F+1}}{2} m_1} \leq K_{\gamma_1,...,\gamma_s;\alpha}[c_{k,\gamma_1}]^{|\gamma_1|}...[c_{k,\gamma_s}]^{|\gamma_s|} \tag{26}$$

2. Positive monotone functions $p_i \in C^\infty(I\!R_+)$ of polynomial behaviour

$$\exists \varepsilon > 0 \ \forall z \in I\!R_+ \ \ p_i(z) \geq \varepsilon \ \ p_i'(z) \geq \varepsilon$$

$$\exists C \ \ (1+z)|p_i''(z)| \leq C p_i'(z) \quad (1+z)p_i'(z) \leq C p_i(z) \tag{27}$$

satisfy condition

$$\exists K_p \ \forall j \in \{2,...,n\} \ \forall i_1,...,i_s, \ s \geq 2 \ \ i_1+...+i_s = j$$

$$[p_j(z)]^j z^{\frac{\mathbf{k}_{F+1}}{2} m_1} \leq K_p [p_{i_1}(z)]^{i_1} ... [p_{i_s}(z)]^{i_s}, \qquad z \in I\!R_+ \tag{28}$$

Theorem 2. *Let F, B satisfy conditions (5)-(9) and y^0, y_τ be solutions to (4) and (21) for $x^0 \in \ell_2(a)$ and zero-one initial data $\widetilde{x}_\gamma$ (23). Suppose that hierarchies (26) and (28) are valid.*

Then the nonlinear quasi-contractive estimate holds

$$\exists M = M_\tau \ \ \forall t \geq 0 \qquad \rho_\tau(y;t) \leq e^{Mt} \rho_\tau(y;0) \tag{29}$$

3. $\ell_p(c)$-valued stochastic integrals and construction of diffusion process and its variations.

In the following Lemma we construct $\ell_p(c)$-valued stochastic integral, appearing in (21), and prove Ito formula for the norm of $\ell_p(c)$-valued continuous processes. This result will permit to work correctly with variations y_τ in $\ell_{m_\tau}(c_\tau)$ scales, arising in nonlinear expression (25).

Lemma 1. *Let $\Phi(t), \ \Psi(t)$ be $\mathcal{F}_t$-adapted processes with values in $\ell_p(c)$, $c \in I\!P, \ p \geq 1$ such that*

$$\forall q \geq 1, \qquad \sup_{t \in [0,T]} \mathbf{E} \left(\|\Phi(t)\|^q_{\ell_p(c)} + \|\Psi(t)\|^q_{\ell_p(c)} \right) < \infty$$

Then the process, defined by coordinates

$$\eta_k(t) = \eta_k(0) + \int_0^t \Phi_k(s) dW_k(s) + \int_0^t \Psi_k(s) ds$$

for $\eta(0) \in L_q(\Omega, \mathbf{P}, \ell_p(c))$*, belongs to the space of continuous* $\mathcal{F}_t$*-adapted processes, equipped with the norm* $(\mathbf{E} \sup_{t\in[0,T]} \|\cdot\|^q_{\ell_p(c)})^{1/q}$ *and Ito formula is fulfilled*

$$\begin{aligned}
&\|\eta(t)\|^q_{\ell_p(c)} = \|\eta(0)\|^q_{\ell_p(c)} + \\
&+q\int_0^t \|\eta(s)\|^{q-p}_{\ell_p(c)} < \eta^\star(s), \eta(s)\Phi(s)dW(s) >_{\ell_p(c)} + \\
&+q\int_0^t \|\eta(s)\|^{q-p}_{\ell_p(c)} < \eta^\star(s), \eta(s)\Psi(s) + \tfrac{p-1}{2}\Phi^2(s) >_{\ell_p(c)} ds + \\
&+\tfrac{q(q-p)}{2}\int_0^t \|\eta(s)\|^{q-2p}_{\ell_p(c)} \sum_{k\in\mathbb{Z}^d} c_k^2|\eta_k(s)|^{2p-2}\Phi_k^2(s)ds
\end{aligned} \tag{30}$$

where we used notation

$$< \eta^\star, y >_{\ell_p(c)} = \sum_{k\in\mathbb{Z}^d} c_k|\eta_k|^{p-2}y_k \tag{31}$$

Moreover $\forall q \geq p \geq 2$ $\forall T > 0$ $\exists K_{q,T}$ *such that*

$$\mathbf{E} \sup_{t\in[0,T]} \| \int_0^t \Phi(s)dW(s)\|^q_{\ell_p(c)} \leq K_{q,T} \int_0^T \mathbf{E}\,\|\Phi(t)\|^q_{\ell_p(c)}dt \tag{32}$$

Remark 1. First note that the coefficients of diffusion process $B(x_k)$ and $F(x_k)$ are transition invariant. Therefore the required by Lemma 1 inclusions $\{B(x_k)\}_{k\in\mathbb{Z}^d}$, $\{F(x_k)\}_{k\in\mathbb{Z}^d} \in \ell_p(a)$ lead to the requirement $\sum_{k\in\mathbb{Z}^d} a_k < \infty$ on topologies of spaces $\ell_p(a)$, where the initial diffusion process (4) can be constructed.

On the contrary, we do not have restrictions on the weights in spaces $\ell_p(c)$ for variational processes y_τ. Indeed, the principal part of variational equations has form $\{B'(x_k)y_{k,\tau}\}_{k\in\mathbb{Z}^d}$, $\{F'(x_k)y_{k,\tau}\}_{k\in\mathbb{Z}^d}$, i.e. has additional factor y_τ. Due to the zero-one initial data for variational equations (23), there is an inclusion $y_\tau(0) \in \ell_p(c)$ for any $c \in \mathbb{P}$. Therefore, it becomes possible to construct variations in any space $\ell_p(c)$.

This is also important for the study of regularity properties of semigroup, because in Lemma 2 we need the estimates on variations, which grow exponentially fast $c_k \sim e^{a|k|}$, $k \in \mathbb{Z}^d$.

Proof. First of all note that for any vector $h \in \ell_p(c)$, $c \in I\!\!P$ the process $\{h_k W_k(t,\omega)\}_{k\in\mathbb{Z}^d} = hW(t,\omega)$ has $\mathbf{P}$ a.e. $\omega \in \Omega$ $\ell_p(c)$-valued continuous on $t \in [0,\infty)$ paths. This fact follows from the Kolmogorov theorem and estimates

$$\mathbf{E}\,\|hW(t)\|^q_{\ell_p(c)} \le (\sum_{k\in\mathbb{Z}^d} c_k h_k^p)^{(q-p)/p}\mathbf{E}\,(\sum_{k\in\mathbb{Z}^d} c_k h_k^p |W_k(t)|^q) =$$

$$= \|h\|^q_{\ell_p(c)} t^{q/2}\mathbf{E}\,|W_0(1)|^q < \infty$$

$$\mathbf{E}\,\|h(W(t)-W(s))\|^q_{\ell_p(c)} = \mathbf{E}\,(\sum_{k\in\mathbb{Z}^d} c_k h_k^p |W(t)-W(s)|^p)^{q/p} \le$$

$$\le (\sum_{k\in\mathbb{Z}^d} c_k h_k^p)^{(q-p)/p}\mathbf{E}\,(\sum_{k\in\mathbb{Z}^d} c_k h_k^p |W_k(t)-W_k(s)|^q) =$$

$$= \|h\|^q_{\ell_p(c)} (t-s)^{q/2}\mathbf{E}\,|W_0(1)|^q < \infty$$

where we used Hölder inequality and the properties of cylinder Wiener process, W_0 is a Wiener process at point $0 \in \mathbb{Z}^d$ of lattice.

Now consider the $\mathcal{F}_t$-adapted process

$$\widetilde{H}(t) = H^i, \quad \text{for} \quad t \in (t_i, t_{i+1}], \; i \ge 0, \quad \text{and} \quad \widetilde{H}(t_0) = H^0 \quad \text{for} \quad t_0 = 0,$$

where all H^i are $\mathcal{F}_{t_i}$-measurable and $H^i \in L_\infty(\Omega,\mathbf{P};\ell_p(c))$. Then due to the continuity of terms $H^i(\omega)(W(t,\omega)-W(t_i,\omega))$ the stochastic integral, defined by

$$\widetilde{Z}_k(t) = \{\textstyle\int_0^t \widetilde{H}(s)dW(s)\}_k =$$

$$= \sum_{j=0}^{i-1} H_k^j (W_k(t_{j+1}) - W_k(t_j)) + H_k^i (W_k(t) - W_k(t_i)), \quad t \in (t_i, t_{i+1}]$$

and $\widetilde{Z}_k(0) = 0$ has $\ell_p(c)$ pathwise continuous version and is a martingale.

Therefore for $\ell_p(c)$-valued continuous martingale $\widetilde{Z}(t)$ due to [8, Th.3.8] we have inequality

$$\mathbf{E} \sup_{t\in[0,T]} \|\widetilde{Z}(t)\|^q \le (\frac{q}{q-1})^q \sup_{t\in[0,T]} \mathbf{E}\,\|\widetilde{Z}(t)\|^q \tag{33}$$

where the r.h.s. norm is finite by assumptions on $H^i \in L_\infty$.

By Ito formula for $f(\widetilde{Z}(t)) = \|\widetilde{Z}(t)\|^q_{\ell_p(c)}$

$$f(\widetilde{Z}(t)) = f(\widetilde{Z}(0)) + q\int_0^t \|\widetilde{Z}(s)\|^{q-p}_{\ell_p(c)} \sum_{k\in\mathbb{Z}^d} c_k|\widetilde{Z}_k(s)|^{p-1}\widetilde{H}_k(s)dW_k(s) +$$

$$+\frac{q(p-1)}{2}\int_0^t \|\widetilde{Z}(s)\|^{q-p}_{\ell_p(c)} \sum_{k\in\mathbb{Z}^d} c_k|\widetilde{Z}_k(s)|^{p-2}\widetilde{H}_k^2 ds +$$

$$+\frac{q(q-p)}{2}\int_0^t \|\widetilde{Z}(s)\|^{q-2p}_{\ell_p(c)} \sum_{k\in\mathbb{Z}^d} c_k^2|\widetilde{Z}_k(s)|^{2(p-1)}\widetilde{H}_k^2(s) ds$$

and due to $\sum |d_k b_k| \leq \sum |d_k| \sum |b_k|$ one has

$$\mathbf{E}\,\|\widetilde{Z}(t)\|^q_{\ell_p(c)} \leq \frac{q(q-2)}{2}\mathbf{E}\int_0^t \|\widetilde{Z}(s)\|^{q-2}_{\ell_p(c)}\|\widetilde{H}(s)\|^2_{\ell_p(c)} ds$$

Finally, using (33), we obtain

$$\mathbf{E}\sup_{t\in[0,T]} \|\widetilde{Z}(t)\|^q_{\ell_p(c)} \leq$$

$$\leq (\tfrac{q}{q-1})^q \tfrac{q(q-1)}{2}\mathbf{E}\sup_{t\in[0,T]} \|\widetilde{Z}(t)\|^{q-2}_{\ell_p(c)} \textstyle\int_0^T \|\widetilde{H}(s)\|^2_{\ell_p(c)} ds \leq$$

$$\leq K_q(\mathbf{E}\sup_{t\in[0,T]} \|\widetilde{Z}(t)\|^q_{\ell_p(c)})^{(q-2)/q}(\mathbf{E}\,(\textstyle\int_0^T \|\widetilde{H}(s)\|^2_{\ell_p(c)} ds)^{q/2})^{2/q}$$

This leads to

$$\mathbf{E}\sup_{t\in[0,T]} \|\widetilde{Z}(t)\|^q_{\ell_p(c)} \leq K_q^{q/2}\mathbf{E}\,(\textstyle\int_0^T \|\widetilde{H}(s)\|^2_{\ell_p(c)} ds)^{q/2} \leq$$

$$\leq K_q^{q/2} T^{(q-2)/q} \textstyle\int_0^T \mathbf{E}\,\|\widetilde{H}(s)\|^q_{\ell_p(c)} ds$$

and gives the statement of theorem for all functions of $\widetilde{H}$ type. Due to their density, closing inequality (32) we have the definition of stochastic integral and inequality (32) for all Φ. Moreover, the martingale property of $Z(t)$ and its $\mathbf{P}$ a.e. continuity is a simple consequence of estimate (32)

$$\mathbf{E}\sup_{t\in[0,T]} \|\int_0^t \widetilde{H_1}dW - \int_0^t \widetilde{H_2}dW\|^q_{\ell_p(c)} \leq K_{q,T}\int_0^T \mathbf{E}\,\|\widetilde{H_1} - \widetilde{H_2}\|^q_{\ell_p(c)} dt$$

which gives uniform on $[0,T]$ convergence on measure and therefore $\mathbf{P}$ a.e. convergence on subsequence.

To prove Ito formula, first note that

$$|\eta_k(s)|^p = |\eta_k(0)|^p + p\int_0^t |\eta_k(s)|^{p-1}\{\Phi_k(s)dW_k(s)+$$

$$+\Psi_k(s)ds\} + \tfrac{p(p-1)}{2}\int_0^t |\eta_k(s)|^{p-2}\Phi_k^2(s)ds$$

Summing up on $k \in \mathbb{Z}^d$ with weights c_k we have Ito formula for $\|\eta_k(t)\|^p_{\ell_p(c)}$ which immediately gives (30). □

Theorem 3. *For $x^0 \in \ell_{p(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon > 0$, $p \geq 2$, equation (4) has a unique strong solution, i.e. $\mathcal{F}_t$-adapted continuous $\ell_p(a)$-valued process y^0, which satisfies (4) in the sense of $(\mathbf{E}\sup_{t\in[0,T]}\|\cdot\|^q_{\ell_p(a)})^{1/q}$ topology, $q \geq 2$. It admits a representation as a sum of $\ell_p(a)$-valued continuous martingale $M_0(t) = \int_0^t B(y^0)dW$ and $\ell_p(a)$-valued continuous finite variation process $V_0(t) = -\int_0^t (F(y^0)+Ay^0)ds$ and fulfills estimate*

$$\forall q \geq 2 \qquad \sup_{t\in[0,T]} \mathbf{E}\,\|y^0\|^q_{\ell_{p(\mathbf{k}_F+1)}(a)} < \infty \tag{34}$$

For $x^0 \in \ell_p(a)$ there is a unique generalized solution $y^0(t,x^0)$, i.e. a limit of strong solutions in the sense of $(\sup_{t\in[0,T]}\mathbf{E}\,\|\cdot\|^q_{\ell_p(a)})^{1/q}$ topology, $q \geq 2$ and the following estimate holds

$$\forall q\ \exists C_{q,p}, D_{q,p} : \quad \sup_{t\in[0,T]} \mathbf{E}\,\|y^0(t,x^0)\|^q_{\ell_p(a)} \leq e^{C_{q,p}T}(\|x^0\|^q_{\ell_p(a)} + D_{q,p}) \tag{35}$$

Moreover

$$\exists C'_{q,p}\ \forall x^0, y^0 \in \ell_p(a) : \quad \sup_{t\in[0,T]} \mathbf{E}\,\|y^0(t,x^0) - y^0(t,y^0)\|^q_{\ell_p(a)} \leq e^{C'_{q,p}T}\|x^0 - y^0\|^q_{\ell_p(a)} \tag{36}$$

Remark, that the construction of solution $y^0(t,x^0)$ in the $\ell_p(a)$, $p \geq 2$ spaces is required for the proof of differentiability with respect to the initial data.

Proof is quite standard. It uses some infinite-dimensional Lipschitz approximations of equation (4) with a successive application of monotone methods, like in [10, 11]. Being a little technical result, it is ommitted. □

Theorem 4. *Let $m_1 > |\tau|$, $m_\gamma = m_1/|\gamma|$ and vectors $\{c_\tau\} \subset I\!\!P$ fulfill (26). Then $\forall x^0 \in \ell_2(a)$ and zero-one initial data $\widetilde{x}_\gamma$ (23) the equation (21) has a unique strong solution y_τ in space $\ell_{m_\tau}(c_\tau)$, i.e. there is $\mathcal{F}_t$-adapted $\ell_{m_\tau}(c_\tau)$-valued continuous process $y_\tau(t, x^0; \widetilde{x}_\gamma, \gamma \subset \tau)$ such that it fulfills equation (21) in the sense of $(\mathbf{E} \sup_{t\in[0,T]} \|\cdot\|^q_{\ell_{m_\tau}(c_\tau)})^{1/q}$ topology, $q \geq m_\tau$.*

It is represented as a sum of $\ell_{m_\tau}(c_\tau)$ continuous martingale $M_\tau(t) = \int_0^t (B'(y^0)y_\tau + \varphi^B_\tau)dW$ and $\ell_{m_\tau}(c_\tau)$ continuous finite variation process $V_0(t) = -\int_0^t (F'(y^0)y_\tau + Ay_\tau + \varphi^F_\tau)ds$. Moreover, the following estimate holds: $\forall q \geq m_\tau$ $\forall R > 0$ $\exists K_\tau(R)$ such that

$$\sup_{t\in[0,T]} \mathbf{E}\, \|y_\tau(t, x^0; \widetilde{x}_\gamma, \gamma \subset \tau)\|^q_{\ell_{m_\tau}(c_\tau)} \leq K_\tau(R) \tag{37}$$

for $R = \max(\|x^0\|_{\ell_2(a)}; \|\widetilde{x}_\gamma\|_{\ell_{m_\gamma}(c_\gamma)},\ \gamma \subset \tau)$.

Proof. The solvability of equations (21) is obtained inductively with respect to the number of points in set $\tau = \{j_1, ..., j_m\}$, $j_i \in Z\!\!\!Z^d$. First of all note that at $|\tau| = 1$ the inhomogeneous parts $\varphi^B_\tau \equiv \varphi^F_\tau \equiv 0$ and the proof of inductive base coincides with the proof of inductive step.

We prove more general result: if for any $\gamma \subset \tau$, $|\gamma| < |\tau|$ the statement of Theorem 4 holds in scale $\{\ell_{m_\gamma}(d^i c_\gamma)\}_{\gamma\subset\tau}$ for any $i \geq 0$, then the same is true for τ. Vector $d \in I\!\!P$ is such that $d_k \geq a_k^{-(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)m_1}$ for some $\varepsilon > 0$.

Introduce notations $F'_\lambda(x) = \lambda(x)F'(x)$ and $B'_\lambda(x) = \lambda(x)B'(x)$ for $\lambda \in C^\infty(I\!\!R^1, [0, 1])$ such that for some $N_\lambda > 0$

$$\lambda(x) = 0 \ \text{ for } \ |x| \geq N_\lambda + 1 \ \text{ and } \ \lambda(x) = 1 \ \text{ for } \ |x| \leq N_\lambda \tag{38}$$

and consider the approximating equation to (21)

$$\begin{aligned} y^\lambda_{k,\tau}(t) = \widetilde{x}_{k,\tau} + \int_0^t \{B'_\lambda(y^0_k)y^\lambda_{k,\tau} + \varphi^B_{k,\tau}\}dW_k - \\ - \int_0^t \{F'_\lambda(y^0_k)y^\lambda_{k,\tau} + (Ay^\lambda_\tau)_k + \varphi^F_{k,\tau}\}ds \end{aligned} \tag{39}$$

Remark that hierarchy (26) holds for vectors $\{d^i c_\gamma\}$ at any fixed $i \geq 0$ and that the zero-one initial data $\widetilde{x}_\gamma \in \ell_{m_\gamma}(d^i c_\gamma)$ at any $i \geq 0$.

Step 1. *Equation (39) has a unique strong solution y^λ_τ in space $\ell_{m_\tau}(d^i c_\tau)$, i.e. there is $\mathcal{F}_t$-adapted $\ell_{m_\tau}(d^i c_\tau)$-valued pathwise continuous process*

$$y^\lambda_\tau(t, x^0; \widetilde{x}_\gamma, \gamma \subset \tau)$$

such that it fulfills equation (39) in the sense of $(\mathbf{E} \sup_{t\in[0,T]} \|\cdot\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q}$ *- topology, $q \geq m_\tau$, and admits a representation as a sum of continuous martingale $M^\lambda_\tau(t) = \int_0^t \{B'_\lambda(y^0) y^\lambda_\tau + \varphi^B_\tau\} dW$ and continuous finite variation process $V^\lambda_\tau(t) = -\int_0^t \{F'_\lambda(y^0) y^\lambda_\tau + A y^\lambda_\tau + \varphi^F_\tau\} ds$.*

Indeed, in the Banach space of $\mathcal{F}_t$-adapted $\ell_{m_\tau}(d^i c_\tau)$-valued pathwise continuous processes $\eta(t)$ equipped with a norm

$$\|\eta\|_{\tau,i} = (\mathbf{E} \sup_{t\in[0,T]} \|\eta(t)\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q}$$

introduce a map

$$(\mathcal{U}\eta)_k(t) = \widetilde{x}_{k,\tau} + \int_0^t \varphi^B_{k,\tau} dW_k - \int_0^t \varphi^F_{k,\tau} ds + \tag{40}$$

$$+ \int_0^t B'_\lambda(y^0_k)\eta_k(s) dW_k(s) - \int_0^t \{F'_\lambda(y^0_k)\eta_k(s) + (A\eta)_k(s)\} ds$$

By Lemma 1 and due to the boundedness of coefficients F'_λ, B'_λ and
$\|A\|_{\mathcal{L}(\ell_{m_\tau}(d^i c_\tau))} < \infty$ we have

$$\rho_T(\mathcal{U}\eta^1, \mathcal{U}\eta^2) \equiv \mathbf{E} \sup_{t\in[0,T]} \|\mathcal{U}\eta^1 - \mathcal{U}\eta^2\|^q_{\ell_{m_\tau}(d^i c_\tau)} \leq$$

$$\leq M_{\tau,\lambda,T} \int_0^T \mathbf{E}\, \|\eta^1(s) - \eta^2(s)\|^q_{\ell_{m_\tau}(d^i c_\tau)} ds \leq M_{\tau,\lambda,T} \int_0^T \rho_s(\eta^1, \eta^2) ds$$

Therefore $\rho_T(\mathcal{U}^m\eta^1, \mathcal{U}^m\eta^2) \leq \dfrac{M^m_{\tau,\lambda,T}}{m!} T^m \rho_T(\eta^1, \eta^2)$ and there is m_0 such that the map $\mathcal{U}^{m_0}$ is a strict contraction in $\|\cdot\|_{\tau,i}$. For $\eta_0 \equiv 0$ by Lemma 1 we have

$$\|\mathcal{U}\eta_0\|_{\tau,i} \leq \|\widetilde{x}_\tau\|_{\ell_{m_\tau}(d^i c_\tau)} +$$

$$+ C_1 \sup_{t\in[0,T]} (\mathbf{E}\, \|\varphi^B_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q} + C_2 \sup_{t\in[0,T]} (\mathbf{E}\, \|\varphi^F_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q}$$

Above we used inequality

$$\left[\mathbf{E}\,(\int_0^T \|Z_s\| ds)^q\right]^{1/q} \le$$
$$\le T^{(q-1)/q}\left(\mathbf{E}\,\int_0^T \|Z_s\|^q ds\right)^{1/q} \le T\left(\sup_{t\in[0,T]} \mathbf{E}\,\|Z_t\|^q\right)^{1/q} \tag{41}$$

for any $\mathcal{F}_t$-adapted Banach space valued process Z_t.

By [2, Theorem 4.15] with $Q(\cdot) = F^{(s)}(\cdot)$ or $B^{(s)}(\cdot)$, $\zeta^0 = \zeta_{\gamma_1} = ... = \zeta_{\gamma_s} = 0$, $s = \ell$ and Hölder inequality with $r_i = \frac{|\tau|+1}{|\gamma_i|}$, $i = 1, ..., s$, $r_0 = |\tau| + 1$ imply for $\varphi^D = \varphi^F$ or φ^B (22)

$$(\sup_{t\in[0,T]} \mathbf{E}\,\|\varphi^D_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q} \le$$

$$\le K \sum_{\gamma_1,\dots,\gamma_s} (\sup_{t\in[0,T]} \mathbf{E}\,(1 + \|y^0\|_{\ell_2(a)})^{q(\mathbf{k}_F+1)r_0})^{1/qr_0} \times \tag{42}$$

$$\times \prod_{j=1}^{s} (\sup_{t\in[0,T]} \mathbf{E}\,(1 + \|y_{\gamma_j}\|_{\ell_{m_{\gamma_j}}(d^i c_{\gamma_j})})^{qr_j})^{1/qr_j}$$

which gives $\|\mathcal{U}\eta_0\|_{\tau,i} < \infty$ by (35) and inductive assumption. Therefore the sequence $\{\mathcal{U}^m\eta_0\}_{m\ge1}$ converges in $\|\cdot\|_{\tau,i}$ to some $\mathcal{F}_t$-adapted $\ell_{m_\tau}(d^i c_\tau)$-valued pathwise continuous process y^λ_τ. By Lemma 1 sequence (40) converges to (39) with corresponding martingale and finite variation parts.

Step 2. $\forall i \ge 0$ $\forall q \ge 1$ $\exists C_\tau$ *such that*

$$\sup_\lambda \sup_{t\in[0,T]} \mathbf{E}\,\|y^\lambda_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)} \le C_\tau \tag{43}$$

where supremum is taken over all functions $\lambda \in C^\infty(I\!R^1, [0,1])$, *which fulfill (38).*

Indeed, by Ito formula for $q \ge 2m_\tau$

$$h(t) = \mathbf{E}\,\|y^\lambda_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)} = h(0)+$$

$$+q\int_0^t \mathbf{E}\,\|y^\lambda_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} < (y^\lambda_\tau)^\star, y^\lambda_\tau(-F'_\lambda y^\lambda_\tau - Ay^\lambda_\tau - \varphi^F_\tau) >_{\ell_{m_\tau}(d^i c_\tau)} ds+$$

$$+\frac{q(m_\tau-1)}{2}\int_0^t \mathbf{E}\,\|y^\lambda_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} < (y^\lambda_\tau)^\star, (B'_\lambda y^\lambda_\tau + \varphi^B_\tau)^2 >_{\ell_{m_\tau}(d^i c_\tau)} ds+$$

$$+\frac{q(q-m_\tau)}{2}\int_0^t \mathbf{E}\,\|y^\lambda_\tau\|^{q-2m_\tau}_{\ell_{m_\tau}(d^i c_\tau)} \sum_{k\in\mathbb{Z}^d} d_k^{2i} c^2_{k,\tau} |y^\lambda_{k,\tau}|^{2(m_\tau-1)} (B'_\lambda y^\lambda_{k,\tau} + \varphi^B_{k,\tau})^2 ds$$

Inequality (7) and property $0\le\lambda(\cdot)\le 1$ give that $\forall M\ \exists K_M$

$$-F'_\lambda(x) + M[B'_\lambda(x)]^2 = -\lambda(x)F'(x) + M\lambda^2(x)[B'(x)]^2 \le$$

$$\le -\lambda(x)F'(x) + M\lambda(x)[B'(x)]^2 \le \lambda(x)K_M \le K_M$$

Using boundedness of $\|A\|_{\mathcal{L}(\ell_{m_\tau}(d^i c_\tau))}$ and inequalities

$$\sum |u_k v_k| \le \sum |u_k| \sum |v_k|,\ |x|^{m-p}|y|^p \le \frac{m-p}{m}|x|^m + \frac{p}{m}|y|^m \quad (44)$$

$$|<\zeta^\star, xy>_{\ell_m(c)}| \le \frac{m-2}{m}\|\zeta\|^m_{\ell_m(c)} + \frac{1}{m}\|x\|^m_{\ell_m(c)} + \frac{1}{m}\|y\|^m_{\ell_m(c)}$$

we obtain

$$h(t) \le h(0) + (q\|A\| + qK_{q-1} + (q-1)^2)\int_0^t h(s)ds +$$

$$+\int_0^t \mathbf{E}\,\|\varphi^F_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)} ds + 2(q-1)\int_0^t \mathbf{E}\,\|\varphi^B_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)} ds \qquad (45)$$

For inductive base $\varphi^F_\tau \equiv \varphi^B_\tau \equiv 0$, $|\tau| = 1$, therefore by Gronwall-Bellmann inequality the statement of Step 2 holds for any $i \ge 0$. Inductive assumption (37) in any $\ell_{m_\gamma}(d^i c_\gamma)$, $|\gamma| < |\tau|$, (35) and (42) give the boundedness of the last two terms in (45). Then the application of Gronwall-Bellmann inequality finishes the proof of (43).

Step 3. $\forall i \ge 0\ \forall q \ge 1$ *for functions* λ, μ *which fulfill (38) we have*

$$\sup_{t\in[0,T]} \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)} \to 0, \qquad N_\lambda, N_\mu \to \infty \qquad (46)$$

Like in Step 2 by Ito formula for $q \geq 2m_\tau$

$$h(t) = \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^q_{\ell_{m_\tau}(d^ic_\tau)} = -q\int_0^t \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\times$$

$$\times\langle (y^\lambda_\tau - y^\mu_\tau)^\star, (y^\lambda_\tau - y^\mu_\tau)\{F'_\lambda y^\lambda_\tau - F'_\mu y^\mu_\tau + A(y^\lambda_\tau - y^\mu_\tau)\}\rangle ds+$$

$$+\tfrac{q(m_\tau-1)}{2}\int_0^t \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\times$$

$$\times\langle (y^\lambda_\tau - y^\mu_\tau)^\star, (B'_\lambda y^\lambda_\tau - B'_\mu y^\mu_\tau)^2\rangle_{\ell_{m_\tau}(d^ic_\tau)} ds+$$

$$+\tfrac{q(q-m_\tau)}{2}\int_0^t \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^{q-2m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\times$$

$$\times\sum_{k\in\mathbb{Z}^d} d_k^{2i}c^2_{k,\tau}|y^\lambda_{k,\tau} - y^\mu_{k,\tau}|^{2(m_\tau-1)}(B'_\lambda y^\lambda_{k,\tau} - B'_\mu y^\mu_{k,\tau})^2 ds$$

Using inequalities (44) and coordinate relations

$$F'_\lambda(y^0)y^\lambda_\tau - F'_\mu(y^0)y^\mu_\tau = (\lambda(y^0)-\mu(y^0))F'(y^0)y^\lambda_\tau + \mu(y^0)F'(y^0)(y^\lambda_\tau - y^\mu_\tau)$$

$$(B'_\lambda(y^0)y^\lambda_\tau - B'_\mu(y^0)y^\mu_\tau)^2 \leq$$

$$\leq 2\mu^2(y^0)[B'(y^0)]^2(y^\lambda_\tau - y^\mu_\tau)^2 + 2(\lambda(y^0)-\mu(y^0))^2[B'(y^0)]^2(y^\lambda_\tau)^2 \leq$$

$$\leq 2\mu(y^0)[B'(y^0)]^2(y^\lambda_\tau - y^\mu_\tau)^2 + 2(\lambda(y^0)-\mu(y^0))^2[B'(y^0)]^2(y^\lambda_\tau)^2$$

we obtain

$$h(t) \leq (q\|A\| + (q-1)^2)\int_0^t h(s)ds + q\int_0^t \mathbf{E}\,\|y^\lambda_\tau - y^\mu_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\times$$

$$\times\langle (y^\lambda_\tau - y^\mu_\tau)^\star, (y^\lambda_\tau - y^\mu_\tau)^2\mu(y^0)\{-F'(y^0) + (q-1)[B'(y^0)]^2\}\rangle + ds$$

$$+\int_0^t \mathbf{E}\,\|(\lambda(y^0)-\mu(y^0))F'(y^0)y^\lambda_\tau\|^q_{\ell_{m_\tau}(d^ic_\tau)} ds+ \tag{47}$$

$$+2(q-1)\int_0^t \mathbf{E}\,\|(\lambda(y^0)-\mu(y^0))B'(y^0)y^\lambda_\tau\|^q_{\ell_{m_\tau}(d^ic_\tau)} ds \tag{48}$$

Due to conditions (8)-(9) for $0 \le \lambda(\cdot) \le \mu(\cdot) \le 1$

$$\begin{gathered}
|F'(y_k^0)(\lambda(y_k^0)-\mu(y_k^0))| \le K\chi\{|y_k^0| \ge N_\lambda\}(1+|y_k^0|^2)^{\frac{\mathbf{k}_{F+1}}{2}} \le \\
\le K a_k^{-(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)} a_k^{\varepsilon}\chi^{2\varepsilon}\{|y_k^0| \ge N_\lambda\}(a_k + a_k|y_k^0|^2)^{\frac{\mathbf{k}_{F+1}}{2}} \le \\
\le K a_k^{-(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)} [a_k \tfrac{|y_k^0|^2}{N_\lambda^2}]^{\varepsilon}(1+\|y^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{F+1}}{2}} \le \\
\le \frac{K a_k^{-(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)}}{N_\lambda^{2\varepsilon}}(1+\|y^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}_{F+1}}{2}+\varepsilon}
\end{gathered} \tag{49}$$

where $\chi\{A\}$ denotes the characteristic function of set A.

Therefore for $d_k \ge a_k^{-(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)m_\tau}$ we have estimate on (47)

$$\begin{gathered}
\sup_{t\in[0,T]} \mathbf{E}\,\|(\lambda(y^0)-\mu(y^0))F'(y^0)y_\tau^\lambda\|^q_{\ell_{m_\tau}(d^i c_\tau)} \le \\
\le \tfrac{1}{N_\lambda^{2\varepsilon q}} K^q \sup_{t\in[0,T]} \mathbf{E}\,(1+\|y^0\|^2_{\ell_2(a)})^{(\frac{\mathbf{k}_{F+1}}{2}+\varepsilon)q}\|y_\tau^\lambda\|^q_{\ell_{m_\tau}(d^{i+1}c_\tau)} \to 0, \\
N_\lambda, N_\mu \to \infty
\end{gathered} \tag{50}$$

where we applied (35) and statement of Step 2. The analogous convergence holds for term (48). Using $0 \le \mu(\cdot) \le 1$ and (7) we have

$$h(t) \le (q\|A\| + qK_{q-1} + (q-1)^2)\int_0^t h(s)ds + \delta_{\lambda,\mu}$$

with $\delta_{\lambda,\mu} \to 0,\ N_\lambda, N_\mu \to \infty$. By Gronwall-Bellmann inequality we obtain (46).

Step 4. *End of the proof: Theorem 4 is fulfilled for y_τ in any space $\ell_{m_\tau}(d^i c_\tau),\ i \ge 0$.*

By Step 3 there is $\mathcal{F}_t$-adapted $\ell_{m_\tau}(d^i c_\tau)$-valued process $y^\#(t, x^0; \widetilde{x}_\gamma, \gamma \subset \tau)$ such that $\forall q \ge m_\tau$

$$\sup_{t\in[0,T]} \mathbf{E}\,\|y_\tau^\# - y_\tau^\lambda\|^q_{\ell_{m_\tau}(d^i c_\tau)} \to 0, \qquad N_\lambda \to \infty \tag{51}$$

To construct the strong solution y_τ it is sufficient to prove that the equation (39) converges to (21) in the topology $(\mathbf{E} \sup_{t\in[0,T]} \|\cdot\|^q_{\ell_{m_\tau}(d^i c_\tau)})^{1/q}$

when $N_\lambda \to \infty$. By Lemma 1 and choice $B'_\lambda(x) = \lambda(x)B'(x)$

$$(\mathbf{E}\sup_{t\in[0,T]} \| \int_0^t \{B'_\lambda(y^0)y^\lambda_\tau - B'(y^0)y^\#_\tau\}dW\|^q_{\ell_{m_\tau}(d^ic_\tau)})^{1/q} \le$$

$$\le K^{1/q}_{q,T}T^{1/q}\sup_{t\in[0,T]}(\mathbf{E}\,\|(\lambda(y^0)-1)B'(y^0)y^\lambda_\tau\|^q_{\ell_{m_\tau}(d^ic_\tau)})^{1/q} \tag{52}$$

$$+K^{1/q}_{q,T}T^{1/q}\sup_{t\in[0,T]}(\mathbf{E}\,\|B'(y^0)(y^\lambda_\tau - y^\#_\tau)\|^q_{\ell_{m_\tau}(d^ic_\tau)})^{1/q} \tag{53}$$

Like in (50) the term (52) tends to zero at $N_\lambda \to \infty$. To the second term we apply [2, Theorem 4.15]

$$(53) \le C\sup_{t\in[0,T]}[\mathbf{E}\,(1+\|y^0\|_{\ell_2(a)})^{q(\mathbf{k}_F+1)}\|y^\lambda_\tau - y^\#_\tau\|^q_{\ell_{m_\tau}(d^{i+1}c_\tau)}]^{1/q} \to 0,$$
$$N_\lambda \to \infty.$$

Above we also used (51) and (35). Therefore the stochastic integral in (39) converges to the stochastic integral in (21) and gives $\ell_{m_\tau}(d^ic_\tau)$-pathwise continuous martingale. The convergence of continuous finite variation part of (39) to the corresponding part of (21) is checked in a similar way.

We obtain, that the r.h.s. of (39) converges in topology $(\mathbf{E}\sup_{t\in[0,T]}\|\cdot\|^q_{\ell_{m_\tau}(d^ic_\tau)})^{1/q}$, thus the l.h.s. y^λ_τ of (39) also has a limit in the same topology: $\exists\, y_\tau$ such that $y^\lambda_\tau \to y_\tau$, $N_\lambda \to \infty$. Such convergence improves (51) and provides a necessary strong solution y_τ as $\ell_{m_\tau}(d^ic_\tau)$ pathwise continuous modification of $y^\#_\tau$.

The uniqueness of strong solution y_τ is proved by induction on $|\tau|$. Suppose that we have shown the uniqueness for all $|\gamma| < |\tau|$. By Ito formula for two different solutions y^1_τ and y^2_τ we have in analogue to Step 3

$$h(t) = \mathbf{E}\,\|y^1_\tau - y^2_\tau\|^q_{\ell_{m_\tau}(d^ic_\tau)} \le q\|A\|\int_0^t h(s)ds+$$

$$+q\int_0^t \mathbf{E}\,\|y^1_\tau - y^2_\tau\|^{q-m_\tau}_{\ell_{m_\tau}(d^ic_\tau)}\sum_{k\in\mathbb{Z}^d} d^i_k c_{k,\tau}|y^1_{k,\tau} - y^2_{k,\tau}|^{m_\tau}\times$$

$$\times\{-F'(y^0_k) + (q-1)[B'(y^0_k)]^2\} \le (q\|A\| + qK_{q-1})\int_0^t h(s)ds$$

where we used (7). By $h(0)=0$ we obtain $h(t)\equiv 0$ which gives the uniqueness.

It remains to show estimate (37). By Ito formula for strong solution y_τ to (21) and by (44)

$$h(t)=\mathbf{E}\,\|y_\tau(t)\|^q_{\ell_{m_\tau}(d^i c_\tau)}\le \|\widetilde{x}_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)}+$$

$$(q\|A\|+(q-1)^2)\int_0^t h(s)ds+q\int_0^t \mathbf{E}\,\|y_\tau(t)\|^{q-m_\tau}_{\ell_{m_\tau}(d^i c_\tau)}\times$$

$$\times\sum_{k\in\mathbb{Z}^d} d^i_k c_{k,\tau}|y_{k,\tau}|^{m_\tau}\{-F'(y^0_k)+(q-1)[B'(y^0_k)]^2\}ds+$$

$$+\int_0^t \mathbf{E}\,\|\varphi^F_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)}ds+2(q-1)\int_0^t \mathbf{E}\,\|\varphi^B_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)}ds$$

We use (35), (7) and inequality (42) to obtain

$$\begin{gathered} h(t)\le \|\widetilde{x}_\tau\|^q_{\ell_{m_\tau}(d^i c_\tau)}+K(R)+ \\ +(q\|A\|+qK_{q-1}+(q-1)^2)\int_0^t h(s)ds \end{gathered} \tag{54}$$

and therefore (37), which ends the proof of Theorem 4. □

4. Nonlinear estimate on variations (Proof of Theorem 2).

First we restrict to the case $x^0\in\ell_{2(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon>0$, i.e. when y^0 is a strong solution in the sense of Theorem 3. Introduce notations

$$h^i_\tau(y;t)=\mathbf{E}\sum_{s=1}^{i}[p_s(z_t)\sum_{\gamma\subset\tau,\ |\gamma|=s}\|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}],\quad i=1,...,|\tau|$$

$$g_\gamma(t)=\mathbf{E}\,p_i(z_t)\|y_\gamma(t)\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)},\quad |\gamma|=i \tag{55}$$

If we prove that for all $\gamma\subset\tau,\ |\gamma|=i$ and $i=1,...,|\tau|$

$$g_\gamma(t)\le e^{D_1 t}g_\gamma(0)+D_2\int_0^t e^{D_1(t-s)}h^{i-1}_\tau(y;s)ds \tag{56}$$

then we will have the recurrence base and step for the statement of Theorem at $i=|\tau|$.

By Ito formula

$$g_\gamma(t)=g_\gamma(0)-\int_0^t \mathbf{E}\,\|y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}(H^{F,B}p_i)(z_s)ds-$$

$$-m_\gamma\int_0^t \mathbf{E}\,p_i(z_s)<y^\star_\gamma, y_\gamma[F'(y^0)y_\gamma+Ay_\gamma+\varphi^F_\gamma]>_{\ell_{m_\gamma}(c_\gamma)}ds+$$

$$+\frac{m_\tau(m_\tau-1)}{2}\int_0^t \mathbf{E}\,p_i(z_s)<y^\star_\gamma,[B'(y^0)y_\gamma+\varphi^B_\gamma]^2>_{\ell_{m_\gamma}(c_\gamma)}ds+$$

$$+2m_\gamma\int_0^t \mathbf{E}\,p_i'(z_s)\sum_{k\in\mathbb{Z}^d}a_kc_{k,\gamma}y^0_kB(y^0_k)|y_{k,\gamma}|^{m_\gamma-2}y_{k,\gamma}\{B'(y^0_k)y_{k,\gamma}+\varphi^B_{k,\gamma}\}ds$$

where we used notation (31) and operator $H^{F,B}$ acts on smooth function $f(\cdot)$ by rule

$$(H^{F,B}f)(x)=\sum_{k\in\mathbb{Z}^d}\{-\frac{1}{2}B^2(x_k)\frac{\partial^2}{\partial x_k^2}+(F(x_k)+(Ax)_k)\frac{\partial}{\partial x_k}\}f(x)$$

Immediately remark that for functions p which fulfills (27) the following property takes place

$$\exists C_1\in\mathbb{R}\quad H^{F,B}p(z)\geq -C_1p(z) \tag{57}$$

for $z=1+\|x\|^2_{\ell_2(a)}$. Indeed,

$$H^{F,B}p(z)=\sum_{k\in\mathbb{Z}^d}a_k\{2F(x_k)x_k-B^2(x_k)-2(Ax)_kx_k\}p'(z)-$$

$$-\sum_{k\in\mathbb{Z}^d}2a_k^2B^2(x_k)x_k^2p_i''(z)\geq -2\|A\|_{\mathcal{L}(\ell_2(a))}zp'(z)+$$

$$+\sum_{k\in\mathbb{Z}^d}a_k\{2F(x_k)x_k-B^2(x_k)\}p'(z)-2z|p_i''(z)|\sum_{k\in\mathbb{Z}^d}a_kB^2(x_k)\geq$$

$$\geq -2\|A\|Cp(z)+\sum_{k\in\mathbb{Z}^d}a_k\{2F(x_k)x_k-(1+2C)B^2(x_k)\}p'(z)\geq$$

$$\geq -2\|A\|Cp(z)+\sum_{k\in\mathbb{Z}^d}a_k\{-K_1x_k^2-K_2\}p'(z)\geq$$

$$\geq -(2\|A\|C+(K_1+K_2)C)p(z)\equiv -C_1p(z)$$

where we successively applied $\sum |u_k v_k| \le \sum |u_k| \sum |v_k|$, (27), (6) and $\sum a_k = 1$.

Using (44) and (57) we obtain

$$g_\gamma(t) \le g_\gamma(0) + (C_1 + m_\gamma \|A\| + (m_\gamma - 1)^2) \int_0^t g_\gamma(s) ds +$$
$$+ m_\gamma \int_0^t \mathbf{E}\, p_i(z_s) \langle y_\gamma^\star, y_\gamma^2 \{-F'(y_k^0) + (m_\gamma - 1)[B'(y_k^0)]^2\} \rangle_{\ell_{m_\gamma}(c_\gamma)} ds +$$
$$+ \int_0^t \mathbf{E} p_i(z_s) \|\varphi_\gamma^F\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds + \tag{58}$$
$$+2(m_\gamma - 1) \int_0^t \mathbf{E} p_i(z_s) \|\varphi_\gamma^B\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds +$$
$$+2 m_\gamma K_4 \int_0^t \mathbf{E}\, z_s p_i'(z_s) < y_\gamma^\star, (1 + [B'(y^0)]^2) y_\gamma^2 >_{\ell_{m_\gamma}(c_\gamma)} ds +$$
$$+2 m_\gamma K_3 \int_0^t \mathbf{E}\, z_s p_i'(z_s) < y_\gamma^\star, (1 + |B'(y^0)|) y_\gamma \varphi_\gamma^B >_{\ell_{m_\gamma}(c_\gamma)} ds$$

Assumption (27), applied to (58), (27) and (7) lead to

$$g_\gamma(t) \le g_\gamma(0) + (C_1 + m_\gamma \|A\| + (m_\gamma - 1)^2 + 2 m_\gamma K_4 C +$$
$$+2 K_3 C (m_\gamma - 1) + m_\gamma K_{m_\gamma - 1 + 2K_4 C}) \int_0^t g_\gamma(s) ds +$$
$$+ \int_0^t \mathbf{E} p_i(z_s) \|\varphi_\gamma^F\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds + 2(m_\gamma - 1) \int_0^t \mathbf{E}\, p_i(z_s) \|\varphi_\gamma^B\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds + \tag{59}$$
$$+2 K_3 C \int_0^t \mathbf{E}\, p_i(z_s) \|(1 + |B'(y^0)|) \varphi_\gamma^B\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds$$

All terms in (59) have the same structure

$$\int_0^t \mathbf{E}\, p_i(z_s) \| \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\ s \ge 2} \mathbf{D}^{\mathbf{s}}(y^0) y_{\alpha_1} \ldots y_{\alpha_s} \|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} ds \tag{60}$$

where function $\mathbf{D}^{\mathbf{s}}(\cdot) = F^{(s)}(\cdot)$, $B^{(s)}(\cdot)$ or $(1 + |B'(\cdot)|) B^{(s)}(\cdot)$. Using condition (8)-(9) and property $2\mathbf{k}_B \le \mathbf{k}_F$ we estimate (60) by

$$(60) \le K_1 \sum_{\substack{\alpha_1 \cup .. \cup \alpha_s = \gamma, \\ s \ge 2}} \int_0^t \mathbf{E}\, p_i(z_s) \times$$
$$\times \sum_{k \in \mathbb{Z}^d} c_{k,\gamma} [\mathbf{D}^{\mathbf{s}}(y_k^0)]^{m_\gamma} |y_{k,\alpha_1}|^{m_\gamma} \ldots |y_{k,\gamma_s}|^{m_\gamma} ds \le$$

$$\leq K_1 \sum_{\dots} \int_0^t \mathbf{E}\, p_i(z_s) \sum_{k\in\mathbb{Z}^d} c_{k,\gamma} a_k^{-\frac{\mathbf{k}_{F+1}}{2} m_\gamma} \times$$

$$\times (a_k + a_k |y_k^0|^2)^{\frac{\mathbf{k}_{F+1}}{2} m_\gamma} |y_{k,\alpha_1}|^{m_\gamma} \dots |y_{k,\alpha_s}|^{m_\gamma} ds \leq \tag{61}$$

$$\leq K_1 \sum_{\substack{\alpha_1\cup..\cup\alpha_s=\gamma,\\ s\geq 2}} \int_0^t \mathbf{E}\, p_i(z_s) z_s^{\frac{\mathbf{k}_{F+1}}{2} m_\gamma} \times$$

$$\times \sum_{k\in\mathbb{Z}^d} c_{k,\gamma} a_k^{-\frac{\mathbf{k}_{F+1}}{2} m_\gamma} |y_{k,\alpha_1}|^{m_\gamma} \dots |y_{k,\alpha_s}|^{m_\gamma} ds$$

By hierarchies (26), (28) we obtain

$$(61) \leq K_1 K_p^{1/|\gamma|} \times$$

$$\times \sum_{\substack{\alpha_1\cup..\cup\alpha_s=\gamma,\\ s\geq 2}} K_{\alpha_1,\dots,\alpha_s;\gamma}^{1/|\gamma|} \int_0^t \mathbf{E} \sum_{k\in\mathbb{Z}^d} \{p_{|\alpha_i|}(z_s) c_{k,\alpha_i} |y_{k,\alpha_i}|^{m_{\alpha_i}}\}^{|\alpha_i|/|\gamma|} ds \leq$$

$$\leq K_1 K_p^{1/|\gamma|} \sum_{\dots} K_{\alpha_1,\dots,\alpha_s;\gamma}^{1/|\gamma|} \sum_{i=1}^{s} \mathbf{E} \int_0^t p_{|\alpha_i|}(z_s) \|y_{\alpha_i}\|_{\ell_{m_{\alpha_i}}(c_{\alpha_i})}^{m_{\alpha_i}} ds \leq$$

$$\leq K_1 K_p^{1/|\gamma|} 2^{|\tau|} \max_{\alpha_1\cup\dots\cup\alpha_s=\gamma\subset\tau} K_{\alpha_1\cup\dots\cup\alpha_s;\gamma}^{1/|\gamma|}\, h_\tau^{i-1}(y;t)$$

Here we used $\forall j = 1, \dots, s$ $|\alpha_i| < |\gamma|$ and inequality $|x_1 \dots x_s| \leq |x_1|^{q_1}/q_1 + \dots + |x_s|^{q_s}/q_s$ with $q_j = |\gamma|/|\alpha_j|$. Finally we have

$$g_\gamma(t) \leq g_\gamma(0) + D_1 \int_0^t g_\gamma(s) ds + D_2 \int_0^t h^{i-1}(y;s) ds$$

which leads to (56) and proves the quasi-contractive nonlinear estimate for $x^0 \in \ell_{2(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon > 0$. The closure to $x^0 \in \ell_2(a)$ is done with application of estimates (36), (62) and polynomiality of p_i. □

5. Regularity of variations and Proof of Theorem 1.

Before the study the differentiability of $y^0(t,x^0)$ on variable x^0 we obtain the continuity of variations with respect to initial data x^0. This result will be applied to close the nonlinear estimate on variations from $x^0 \in \ell_{p(\mathbf{k}+1)^2+\varepsilon}(a)$ to $x^0 \in \ell_2(a)$ and to prove C^∞-differentiability of $y^0_t(x^0)$ with respect to the initial data x^0.

Theorem 5. *Let $m_1 > |\tau|$, $m_\gamma = m_1/|\gamma|$, vectors $\{c_\tau\} \subset I\!\!P$ fulfill (26) and $\widetilde{x}_\gamma$ be zero-one initial data (23). Then $\forall q \ge m_\tau$ $\forall R > 0$ $\exists K_\tau(R)$ such that $\forall x^0, y^0 \in \ell_2(a)$ the variations fulfill*

$$\sup_{t\in[0,T]} \mathbf{E}\,\|y_\tau(t,x^0;\widetilde{x}_\gamma,\gamma\subset\tau) - y_\tau(t,y^0;\widetilde{x}_\gamma,\gamma\subset\tau)\|^q_{\ell_{m_\tau}(c_\tau)} \le$$
$$\le K_\tau(R)\|x^0-y^0\|^q_{\ell_2(a)} \tag{62}$$

with $R = \max(\|x^0\|_{\ell_2(a)}, \|y^0\|_{\ell_2(a)}, \|\widetilde{x}_\gamma\|_{\ell_{m_\gamma}(dc_\gamma)})$ for $d_k \ge a_k^{-\frac{\mathbf{k}_{F+1}}{2}m_1}$, $k \in Z\!\!\!Z^d$.

Proof is similar to the proof of nonlinear estimate on variations and proceeds with application of Ito formula instead of pathwise estimates of [2, Th.4.18]. □

To obtain the integral representation of Theorem 6, we need the following Lemma, which gives uniform on $|\tau| \le n_0$ estimates on variations. This result is also required for the study the high order differentiability of the stochastic flow and heat semigroup P_t.

Lemma 2. *Under conditions (5)-(9) for zero-one initial data $\widetilde{x}_\gamma$ (23) we have*

$$\forall\psi\in I\!\!P\ \forall n\ge 1\ \forall q\ge 1\quad \exists K_n(R,\psi,q)\quad \textit{such that}$$

$$\sup_{t\in[0,T]} \mathbf{E}\,|y_{k,\tau}(t,x^0,\widetilde{x}_\gamma)|^q \le K_n(R,\psi,q)a_k^{-\frac{\mathbf{k}_{F+1}}{2}q(|\tau|-1)} \prod_{j\in\tau}\psi^{-1}_{k-j} \tag{63}$$

$$\sup_{t\in[0,T]} \mathbf{E}\,|y_{k,\tau}(t,x^0,\widetilde{x}_\gamma) - y_{k,\tau}(t,y^0,\widetilde{x}_\gamma)|^q \le$$
$$\le K_n(R,\psi,q)a_k^{-\frac{\mathbf{k}_{F+1}}{2}q(2|\tau|-1)} \prod_{j\in\tau}\psi^{-1}_{k-j}\|x^0-y^0\|^q_{\ell_2(a)} \tag{64}$$

with $R = \max(\|x^0\|_{\ell_2(a)}, \|y^0\|_{\ell_2(a)})$.

Proof uses a special choice of weights $\widetilde{c}_{k,\gamma} = a_k^{\frac{\mathbf{k}_{F+1}}{2} m_1 \frac{|\gamma|-1}{|\gamma|}} \prod_{j\in\gamma} \psi_{k-j}$, $\gamma \subset \tau$ with $m_1 \overset{def}{=} q|\tau|$ and coincides with proof of [2, Corollary 4.19]. It can be omitted. $\square$

Now we turn to the differentiability of process y^0 (4) with respect to the initial data.

Theorem 6. *Let F, B satisfy conditions (5)-(9). Then $\forall x^0 \in \ell_2(a)$, zero-one initial data $\widetilde{x}_\gamma$ (23) and $h \in \mathbf{AC}([a,b])$ for all $t \in [0,T]$ and $\mathbf{P}$ a.e. $\omega \in \Omega$ the path*

$$\chi^0(\cdot) = y^0(t, x^0 + h(\cdot)) - y^0(t, x^0 + h(a)) \in \mathbf{AC}([a,b])$$

In particular, in any space $\ell_p(c),\ c \in I\!\!P,\ p \geq 1$ its derivative is given by first order variation

$$y^0(t, x^0 + h(\cdot))\Bigg|_a^b = \ell_p(c) \int_a^b \sum_{j\in \mathbb{Z}^d} y_{\{j\}}(t, x^0 + h(s)) h_j'(s) ds \qquad (65)$$

Space $\mathbf{AC}([a,b])$ was introduced in (16).

Proof. First we prove representation (65) for initial data $x^0 \in \ell_{m_1(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon > 0$, in space $L_q(\Omega, \mathbf{P}, \ell_{m_1}(c_1)),\ q \geq 1$, with vector $c_1 \in I\!\!P$ such that $d_k c_{k,1} \leq a_k$ for $d_k \geq a_k^{-\frac{\mathbf{k}_{F+1}}{2} m_1}$. Due to Theorem 3 for $x^0 \in \ell_{m_1(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon > 0$, there is a strong solution y^0 to equation (4) in space with topology $\mathbf{E} \sup_{t\in[0,T]} \|\cdot\|^q_{\ell_{m_1}(a)}$ and estimate holds $\mathbf{E}\,\|y^0(t,x^0) - y^0(t,y^0)\|^q_{\ell_{m_1}(a)} \leq e^{C_q t}\|x^0 - y^0\|^q_{\ell_{m_1}(a)}$. Inequality $\|\cdot\|_{\ell_{m_1}(c_1)} \leq \|\cdot\|_{\ell_{m_1}(a)}$ implies that for function $h \in \mathbf{AC}([a,b])$ the map $[a,b] \ni s \to y^0(t, x^0 + h(s)) \in L_q(\Omega, \mathbf{P}, \ell_{m_1}(c_1))$ is absolutely continuous. The theory of absolutely continuous functions in reflexive Banach space gives that for a.e. $s \in [a,b]$ there is $L_q(\Omega, \mathbf{P}, \ell_{m_1}(c_1))$ strong derivative $\frac{d}{ds} y^0(t, x^0 + h(s))$ and representation holds

$$y^0(t, x^0 + h(\cdot))\Bigg|_a^b = L_q(\Omega, \mathbf{P}, \ell_{m_1}(c_1)) \int_a^b \frac{d}{ds} y^0(t, x^0 + h(s)) ds \quad (66)$$

To reconstruct the strong derivative let us show that for $h \in \mathbf{AC}([a,b])$ and a.e $s \in [a,b]$ such that

$$\lim_{\alpha\to 0} \| \frac{h(s+\alpha)-h(s)}{\alpha} - h'(s)\|_{\ell_{m_1}(a)} = 0$$

the convergence holds

$$\sup_{t\in[0,T]} \mathbf{E} \left\| \frac{y_k^0(t,x^0+h(s+\alpha)) - y_k^0(t,x^0+h(s))}{\alpha} - \right.$$

$$\left. - \sum_{j\in \mathbb{Z}^d} y_{k,\{j\}}(t,y^0) h_j'(s) \right\|^q_{\ell_{m_1}(c_1)} \to 0, \quad \alpha \to 0$$

Further proof coincides with the proof of [2, Th.4.20] with use of Ito formula instead of pathwise estimates. □

Next Theorem states any order differentiability of process $y^0(t,x^0)$.

Theorem 7. *Let F, B fulfill conditions (5)-(9). Then $\forall x^0 \in \ell_2(a)$, zero-one initial data $\widetilde{x}_\gamma$ (23) and $h \in \mathbf{AC}([a,b])$ (16) we have for all $t \in [0,T]$, $\mathbf{P}$ a.e. $\omega \in \Omega$ and $\forall k \in \mathbb{Z}^d$, $\forall \tau$ the path*

$$\chi_{k,\tau}(\cdot) = y_{k,\tau}(t,x^0+h(\cdot)) - y_{k,\tau}(t,x^0+h(a)) \in AC([a,b], \mathbb{R}^1)$$

In particular different order variations are related by

$$y_{k,\tau}(t,x^0+h(\cdot))\Big|_a^b = \int_a^b \sum_{j\in \mathbb{Z}^d} y_{k,\tau\cup\{j\}}(t,x^0+h(s)) h_j'(s) ds$$

Proof. Like in the proof of Theorem 6 we first consider initial data $x^0 \in \ell_{m_1(\mathbf{k}_F+1)^2+\varepsilon}(a)$, $\varepsilon > 0$, for some $m_1 > |\tau|$. Choose vectors $\{c_n\}_{n\ge 1}$ so that

$$\forall k \in \mathbb{Z}^d \quad c_{k,n+1} d_k \le c_{k,n}, \qquad c_{k,1} d_k \le a_k \tag{67}$$

with $d_k \ge a_k^{-\frac{\mathbf{k}_{F+1}}{2} m_1}$. These vectors obviously satisfy condition (26).

Introduce notation $X_{|\tau|} = \ell_{m_\tau}(c_{|\tau|})$. Applying Theorem 5 in scale $\{X_{|\tau|}\}$ and inequality $\|\cdot\|_{X_{|\tau|+1}} \le const \|\cdot\|_{X_{|\tau|}}$ we have the absolute continuity of the map

$$[a,b] \ni s \to y_\tau(t,x^0+h(s)) \in L_q(\Omega, \mathbf{P}, X_{|\tau|+1})$$

for any $t \in [0,T]$ and $h \in \mathbf{AC}([a,b])$. The theory of absolutely continuous functions implies the existence of strong derivative

$$L_q(\Omega, \mathbf{P}, X_{|\tau|+1}) \frac{d}{ds} y_\tau(t, x^0 + h(s)) \text{ for a.e. } s \in [a,b]$$

and gives representation

$$y_\tau(t, x^0 + h(\cdot))\Big|_a^b = L_q(\Omega, \mathbf{P}, X_{|\tau|+1}) \int_a^b \frac{d}{ds} y_\tau(t, x^0 + h(s)) ds \quad (68)$$

If we prove by induction on $|\tau|$ that for a.e. $s \in [a,b]$ such that

$$\exists \lim_{\alpha \to 0} \| \frac{h(s+\alpha) - h(s)}{\alpha} - h'(s) \|_{\ell_{m_1}(a)} = 0 \quad (69)$$

the convergence holds

$$\sup_{t \in [0,T]} \mathbf{E} \left\| \frac{y_{k,\tau}(t, x^0 + h(s+\alpha)) - y_{k,\tau}(t, x^0 + h(s))}{\alpha} - \right. \\ \left. - \sum_{j \in \mathbb{Z}^d} y_{k,\tau \cup \{j\}} h'_j(s) \right\|_{X_{|\tau|+1}}^q \to 0 \quad (70)$$

for $\alpha \to 0$, then the representation (68) will lead to

$$y_\tau(t, x^0 + h(\cdot))\Big|_a^b = L_q(\Omega, \mathbf{P}, X_{|\tau|+1}) \int_a^b \sum_{j \in \mathbb{Z}^d} y_{\tau \cup \{j\}}(t, x^0 + h(s)) h'_j(s) ds$$

This gives the $\mathbf{P}$ a.e. coordinate equality: $\forall k \in \mathbb{Z}^d$

$$y_{k,\tau}(t, x^0 + h(\cdot))\Big|_a^b = \int_a^b \sum_{j \in \mathbb{Z}^d} y_{k,\tau \cup \{j\}}(t, x^0 + h(s)) h'_j(s) ds \quad (71)$$

with integrable for $\mathbf{P}$ a.e. $\omega \in \Omega$ right hand side

$$\sum_{j \in \mathbb{Z}^d} y_{k,\tau \cup \{j\}}(t, x^0 + h(\cdot)) h'_j(\cdot) \in L_1([a,b], \mathbb{R}^1) \quad (72)$$

Further proof proceeds similar to [2, Th.4.21], with the use of Ito formula for convergence (70) instead of pathwise estimates. ■

The developed above technique is sufficient for the study of differentiable properties of Feller semigroup P_t (10).

Proof of Theorem 1. It completely coincides with one, conducted in [2, § 4.6] for the unit diffusion case. The only difference is that, using representation

$$\partial_\tau P_t f(x^0) = \sum_{\sigma=1}^{|\tau|} \sum_{\gamma_1 \cup \ldots \cup \gamma_\sigma = \tau} \mathbf{E} < \partial^{(\sigma)} f(y^0), y_{\gamma_1} \otimes \ldots \otimes y_{\gamma_\sigma} > (t, x^0) \quad (73)$$

with variations y_γ (21) and

$$< \partial^{(\sigma)} f(y^0), y_{\gamma_1} \otimes \ldots \otimes y_{\gamma_\sigma} > (t, x^0) =$$

$$= \sum_{j_1,\ldots,j_\sigma \in \mathbb{Z}^d} \partial_{\{j_1,\ldots,j_\sigma\}} f(y^0(t,x^0)) y_{j_1,\gamma_1}(t,x^0) \ldots y_{j_\sigma,\gamma_\sigma}(t,x^0)$$

one should use existence of majorant to show the measurability of derivatives $\partial_\tau P_t f(x)$. □

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