 E. Shcherbakov

AXISYMMETRICAL MIXED FREE BOUNDARY VALUE PROBLEM

Mixed Free Boundary Value Problem for Laplace equation in axisymmetrical case is considered. We take into consideration mean and Gauss curvatures of the free boundary. The problem of this type arise on investigating thermal equillibrium of two phases. We take into account capillary forces acting in intermediate layer separating different phases. Plane model of the equillibrium without capillary forces was considered in the paper([1]). We consider variational problem whose solution is generalized solution of the boundary value problem. We prove regularity of solution, analyticity of free boundary and investigate its properties.

Keywords and phrases: Mixed boundary value problem, mean curvature, Gausss curvature, variational problem, free boundary
MSC (2000): 35R35

1. Boundary Value Problem.

All the domains considered in the paper are meridional sections of axisymmetrical domains.We are searching for a domain Ω, lying in the half strip$\Pi = \{(x,y)\,|x \in (-1,1)\,, y \in (-\infty, 0)\}$which is symmetrical with regard to the axis y. The boundary of Ω consists of the set $\Gamma = \{(\pm 1, y)\,, y \in (-\infty, 0)\}$ and unknown Σ with the endponts $(\pm 1, 0)$. We suppose the curve $\Sigma$to be twice differentiable everywhwere except of the set Λ_Σ of the points lying on the y-axis where it possibly posesses singularity. We are searching also for a function $u = u\,(x,y)$, $u\,(x,y) = u\,(-x,y)$, a solution of the following equation

$$\frac{1}{x} \cdot \frac{\partial}{\partial x}\left(x \cdot \frac{\partial u\,(x,y)}{\partial x}\right) + \frac{\partial^2 u\,(x,y)}{\partial y^2} = 0 \quad (x,y) \in \Omega, x > 0 \quad (1.1)$$

We suppose that it satisfies the following boundary conditions

$$u\,(x,y) = 1, \quad (x,y) \in \Sigma_r, \quad (1.2)$$

$$\frac{\partial u}{\partial \nu}\,(\pm 1, y) + \alpha \cdot u\,(\pm 1, y) = 0, \quad y \in (-\infty, 0)\,, \quad (1.3)$$

$$u(\infty) = 0 \tag{1.4}$$

$$l\frac{1}{2} \cdot |\nabla u|^2 + \kappa \cdot \mathrm{H}(x,y) + \theta \cdot \mathrm{K}(x,y) = \lambda, \qquad (x,y) \in \Sigma_r \bigcap (\Pi - \Lambda_\Sigma), \; x \neq \pm 1. \tag{1.5}$$

Here Σ_r is the set of regular points of the free arcΣ for the Dirichlet boundary value problem (see[2]). The functions $\mathrm{H}(x,y)$, $\mathrm{K}(x,y)$ represent values of mean and Gauss curvature of the axisymmetrical surface S at a point (x,y) of its meridional section Ω. The letter νdenotes external normal to the set Γ. The numbers α, κ, θ are nonnegative real numbers. When function u satisfies Laplace equation, $\kappa = \theta = 0$, we have plane boundary value problem studied in (see[1]). In the sequel we shall call to the problem (1.1)-(1.5) as the main boundary value problem.

2. Variational Problem.

We use the variational method for procuring solution of described free boundary value problem. Let

$$\begin{aligned} I(u,\Omega) \quad := & \iint\limits_{\Omega} |\nabla u(x,y)|^2 \, |x| \; dx\, dy - \lambda \iint\limits_{B} |x| \; dx\, dy \\ & + \kappa \int\limits_{\Sigma} |x| \; ds + 2\theta \int\limits_{\Sigma} f(\dot{x})\, ds + \alpha \int\limits_{\Gamma} u^2 \, ds \end{aligned} \tag{2.1}$$

Here B is complement $\Pi \setminus \bar{\Omega}$ to closed domain $\bar{\Omega}$ in the half strip Π, $z = z(s) = y(s) + ix(s)$ is the natural parametric representation of curve Σ, $\dot{x}(s)$- derivative $\dfrac{dx}{ds}$, and f is the function of the following type:

$$\begin{aligned} 2f(t) = -\sqrt{1-t^2} \cdot \int\limits_0^t (\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}) \times \\ \times (1-\sigma^2)^{-\frac{3}{2}} \, d\sigma + \mathrm{E}_0 \sqrt{1-t^2}. \end{aligned} \tag{2.2}$$

Let us consider the functional I over the set$D$of the admissible pairs (u, Ω) defined as follows. The boundary of the domain Ω consists of rectifiable curves $\Sigma = \delta\Omega \bigcap \Pi$, symmetrical with regard to$y$ axis and of the set Γ of the above mentioned type. It is clear that the curve Σ connecting the points $(\pm 1, 0)$ define admissible domains. We denote as $\Xi$the set of admissible curves Σ.We denote as W the class of admissible functions u,$u\,(x, y) = u\,(-x, y)$. These functions are continuous in the domain$\bar{\Omega} \setminus \Lambda_\Sigma$. They assume boundary value equal to one on the curve Σ, and vanish at the infinite point. We will suppose that the functions $u\,(x, y)$ posess on the setΩ generalized derivatives of the first order such that weighted Dirichlet integral is bounded

$$\iint\limits_{\Omega} |\nabla u(x, y)|^2 \cdot |x| \; dx\, dy < \infty$$

Variational problem: Find minimum of the functional$I = I\,(u, \Omega)$ on the set D.

3. Symmetrization of functions and domains.

Let Ω be an admissible domain, Ω_h $=\Omega \bigcap [(x, y)\,,\ -1 \le x \le 1,$ $y > -h, h > 0]$ and

$$I_0\,(u, \Omega_h) = I_0^{'}\,(u, \Omega_h) + I_0^{''}\,(u, \Omega_h) \tag{3.1}$$

$$I_0^{'}\,(u, \Omega_h) = \iint\limits_{\Omega_h} |\nabla u(x, y)|^2\, |x| \; dx\, dy \tag{3.2}$$

$$I_0^{''}\,(u, \Omega_h) = \alpha \cdot \iint\limits_{\Gamma_h} |u(x, y)|^2\, ds \tag{3.3}$$

$h \le \infty$. In this section we are going to study the behaviour of the functional I_0 under symmetrization of the solution of the boundary value problem (1.1)-(1.4). We start with the following principle of maximum (see[1]).

Lemma 3.1. *Let Σ be analytic curve symmetrical with regard to axis y and $u = u\,(x, y)$ - a solution of the boundary value problem (1.1)-(1.4). Then the function$u$attains its maximum value on the curve$\Sigma$and this value is equal to one.*

Proof. For the function u given we can choose a number h sufficiently large and such that the following inequality takes place $|u(x,-h)| < \frac{1}{2}\cdot \sup\{u(z), z\in\Omega\}$. The function u cannot assume its maximum value inside of the domain Ω. It cannot also have the maximum value on the line Γ. It means that the maximum value can be achieved only on the curve Σ. It is clear that this value is equal to one. The lemma is proved.

Let us now consider symmetrization of the function $u \in W$ with regard to the axis x. We can extend the function under consideration as unity into the complement to the domain $\overline{\Omega}$ to the domain Π. Let us put $t = x^2$, $y = y$ $\nu(t,y) = u\left(\sqrt{t}, y\right), if \geq 0$,and $u\left(\sqrt{|t|}, y\right), if t < 0$. Besides we put ν equal to one at the points where the function $u$assumes the same value. Thus we we have that the function ν is defined everywhere in the half-strip Π. Let

$$G = \{(t,y,z) : (t,y) \in \Pi, z < \nu(t,y)\}$$

We have from the lemma (3.1) that

$$G \subset \{(t,y,z) : (t,y) \subset \Pi, z < 1\}$$

We denote as $G(y_0)$ a section of the domain G by the plane $\{(t,y,z) : y = y_0\}$. It is clear that $\delta G(y_0)$ coincides with the graph of the function$\nu_0(t) := \nu(t,y_0)$. Let us now put

$$\rho_1(y_0) = \inf\{\nu_0(t) : |t| \leq 1\},$$

$$\rho_2(y_0) = \sup\{\nu_0(t) : |t| \leq 1\}.$$

We see that the intersection of $\delta G(y_0)$with the line

$$l(y_0,\rho) = \{(t,y_0,\rho) : |t| \leq 1, \rho_1 \leq \rho \leq \rho_2\}$$

consists of the even number of the points $t_k(\rho)$, $1 \leq k \leq 2n$. Let us denote as $2\cdot T(\rho)$ the measure of the intersection of $G(y_0)$ with the line $l_0(y_0,\rho)$,

$$2\cdot T(\rho) = t_2(\rho) - t_1(\rho) + \cdots + t_{2n}(\rho) - t_{2n-1}(\rho)$$

It is clear that this measure is equal to 2 when $\rho < \rho_1(y_0)$ The function $T(\rho)$thus defined is an increasing function.

Definition 3.1. Let $\nu^\bullet = \nu^\bullet(t,y)$ be the function corresponding to the function ν in the following way

- $\nu^{\bullet}(-t,y) = \nu^{\bullet}(t,y)\,, t \in [0,1],$
- $\nu^{\bullet}(T(\rho)) = \rho, \rho_1(y) \leq \rho \leq \rho_2(y)\,, y \leq\ y_0.$

We call the function $u^{\bullet} = u^{\bullet}(x,y) = \nu^{\bullet}(x^2,y)\,, |x| \leq 1$ the symmetrization of the function u with regard to the axis x. The line Σ is the level line for the function u which means that symmetrization of the function leads to the Steiner symmetrization of the domain Ω with regard to the axis y. We denote as $\Omega^{\bullet}$ the symmetrization of the domain $\Omega$and we select the notation $u^{\bullet}$ for the symmetrization of the function u. Now we are going to define the symmetrization of the function u with regard to the axis x. Once again we extend this function into the half-strip Π by the rule $u(x,y) = u(x,-y)\,, -\infty < y \leq 0$. Let us define as $G^{'}$the domain G with its reflection in the plane (x,y). We denote as $G^{'}(x_0)$the intersection of the domain G with the plane $\{x = x_0\}$. It is clear that the boundary $\delta G^{'}(x_0)$is the graph of the function $u_0(x_0,y)\,, |x_0| < 1$.We can transform this graph in the same way as it was done with the graph of $G(y_0)$ of the function ν_0.As a result we get the function $u^{\bullet\bullet}(t,y)$ whose graph will be symmetric with regard to the plane (x,z). As in preceding case the symmetrization of the function of this type leads us to the symmetrization of the domain $\Omega \bigcup \{(x,y)\,|x| \leq 1, y > 0, (x,-y) \in \Omega\}$ - in this case with regard to the axis$x$. Let us study now the behaviour of the functionalI_0 under symmetrizations of the solutions of the boundary value(1.1)-(1.4) just defined.

Theorem 3.1. *Let $u(x,y)$be a solution of the problem (1.1)-(1.4)in the domain Ω and $u^{\bullet} = u^{\bullet}(x,y)$-its symmetrization with regard to some of its axis. Then*

$$I_0(u^{\bullet},\Omega^{\bullet}) \leq I_0(u,\Omega) \tag{3.4}$$

Proof. For the beginning we consider symmetrization with regard to the axis y. After substitution of the variables u,ν by the variables $u^{\bullet}$, $\nu^{\bullet}$ under the sign of integral I_0 we arrive at the following expression

$$I_0^{'}(u^{\bullet},\Omega^{\bullet}) = 4 \cdot \int_0^1 \int_0^1 t\left(\nu_t^{\bullet 2}\right) \cdot\, dt \cdot\, dy + \int_0^1 \int_0^1 \left(\nu_y^{\bullet 2}\right) \cdot\, dt \cdot\, dy \tag{3.5}$$

Now the function $T = T(\rho)$ is monotone one for each $y_0 \in (-\infty, 0)$. Hence the following representation takes place

$$\int_0^1 \left(\nu_y^\bullet\right)^2 (t, y_0) \cdot dt = \int_{\rho_1(y_0)}^{\rho_2(y_0)} \left[\frac{\left(\frac{\delta T}{\delta y}\right)^2}{\left|\frac{\delta T}{\delta \rho}\right|} \right] (\rho, y_0) \cdot d\rho \tag{3.6}$$

This representation is a plane consequence of the second property from the defintion 3.1(see[4]). The function u is a real analytic function. This means that the graph of the function $\nu_0(t)$ is the union of the monotone curves. For each of this curves we can do calculations which leads us to the formula (3.6). Thus we get as a result the following formula

$$\int_0^1 \left(\nu_y\right)^2 (t, y_0)\, dt = \int_{\rho_1(y_0)}^{\rho_2(y_0)} \sum_{i=1}^{2n(\rho)} \left[\frac{\left(\frac{\delta t_i}{\delta y}\right)^2}{\left|\frac{\delta t_i}{\delta \rho}\right|} \right] (\rho, y_0)\, d\rho \tag{3.7}$$

Now we use the Shwartz inequality to get the following result

$$\left[\sum_{i=1}^{2\cdot n(\rho)} (-1)^i \cdot \frac{\delta t_i}{\delta y}\right]^2 \leq \sum_{i=1}^{2n(\rho)} \frac{\left(\frac{\delta t_i}{\delta y}\right)^2}{\left|\frac{\delta t_i}{\delta \rho}\right|} \cdot \left[\sum_{1=1}^{2n(\rho)} (-1)^i \cdot \frac{\delta t_i}{\delta \rho}\right] =$$

$$= \sum_{i=1}^{2n(\rho)} \left(\frac{\left(\frac{\delta t_i}{\delta y}\right)^2}{\left|\frac{\delta t_i}{\delta \rho}\right|} \right) \left|\frac{\delta T}{\delta \rho}\right|$$

From this inequality we easily get

$$\int_{\rho_1(y_0)}^{\rho_2(y_0)} \left[\frac{\left(\frac{\delta T}{\delta y}\right)^2}{\left|\frac{\delta T}{\delta \rho}\right|} \right] (\rho, y_0)\, d\rho = \int_{\rho_1(y_0)}^{\rho_2(y_0)} \left[\frac{\left(\frac{\delta t_i}{\delta y}\right)^2}{\left|\frac{\delta t_i}{\delta \rho}\right|} \right] (\rho, y_0)\, d\rho \tag{3.8}$$

Comparing the expressions (3.6)-(3.8) we get the following result

$$\int_{-\infty}^{0} \int_0^1 \left(\nu_y{}^{\bullet 2}\right)(t, y)\, dt \cdot dy \leq \int_{-\infty}^{0} \int_0^1 \left(\nu_y{}^{\bullet 2}\right)(t, y)\, dt \cdot dy \tag{3.9}$$

We prove now that there also takes place the following inequality

$$\int_{-\infty}^{0}\int_{0}^{1}\left(\nu_t{}^{\bullet 2}\right)(t,y)\cdot t\,dt\cdot\,dy \le \int_{-\infty}^{0}\int_{0}^{1}\left(\nu_t{}^{\bullet 2}\right)(t,y)\cdot t\,dt\cdot\,dy \qquad (3.10)$$

To this end let us consider the following quiet evident inequality

$$\left[\sum_{i=1}^{2n(\rho)}(-1)^i t_i\right]^2 \le \left[\sum_{i=1}^{2n(\rho)}\frac{t_i}{\left|\frac{\delta t_i}{\delta\rho}\right|}\right]\cdot\left[\sum_{i=1}^{2n(\rho)}(-1)^i\frac{\delta t_i}{\delta\rho}\right] =$$

$$= \left[\sum_{i=1}^{2n(\rho)}\frac{t_i}{\left|\frac{\delta t_i}{\delta\rho}\right|}\right]\cdot\left|\frac{\delta T}{\delta\rho}\right|.$$

We get from it the result

$$\frac{T}{\left|\frac{\delta T}{\delta\rho}\right|} \le \left[\sum_{i=1}^{2n(\rho)}\frac{t_i}{\left|\frac{\delta t_i}{\delta\rho}\right|}\right].$$

It leads us to the inequality

$$\int_{-\infty}^{0}\int_{0}^{1} t\left(\nu_t{}^{\bullet}\right)^2(t,y)\,\,dt\cdot\,dy \le \int_{-\infty}^{0}\int_{\rho_1(y_0)}^{\rho_2(y_0)}\frac{T}{\left|\frac{\delta T}{\delta\rho}\right|}\,d\rho\,dy \le$$

$$\int_{-\infty}^{0}\int_{\rho_1(y_0)}^{\rho_2(y_0)}\sum_{i=1}^{2n(\rho)}\frac{t_i}{\left|\frac{\delta t_i}{\delta\rho}\right|}\,d\rho\,dy = \int_{-\infty}^{0}\int_{0}^{1} t\left(\nu_t{}^2\right)(t,y)\,\,dt\cdot\,dy.$$

It is in fact the inequality (3.10). Now uniting the inequalities (3.9)-(3.10)we arrive at the result

$$\acute{I}_0\left(u^{\bullet},\Omega^{\bullet}\right) \le \acute{I}_0\left(u,\Omega\right) \qquad (3.11)$$

It is now left to prove that

$$\int_{\Gamma} u^{\bullet 2}\,ds \le \int_{\Gamma} u^2\,ds \qquad (3.12)$$

The function $T(\rho)$ is a monototone one. Hence for each fixed $y_0, -\infty \leq y_0 \leq 0$, the function $u^\bullet$ achieves its minimal value on on the line $\{x = \pm 1\}$. It means that on the setΓ the values of the function $u^\bullet$ do not exceed the values of the function u. These arguments lead us to the conclusion that the inequality(3.12) is valid. From the inequalities(3.11)-(3.12)it now follows that under the symmetrization of the function $u$with regard to the axis y the inequality (3.4)takes place. The same result also takes place for the symmetrization with regard to the axis x. This time it will be even more easy to prove the assertion as we will have now in the expression (3.12) equality instead of the inequality. The theorem is proved.

4. Dirichlet Principle.

We will show now that solutions of mixed boundary value problem(1.1)-(1.4) possess extremal property usually called as Dirichlet Principle. Let Ω be an admisssible domain with the curve Σ monotone in each quadrant from lower half-plane. We supppose also that this curve consists of the regular points and is symmetric with regard to the axis y. The aim of this section consists of the proof of the Dirichlet Principle for the solutions of the problem (1.1)–(1.4). This problem is singular because of infinity of the domain. In the paper [1] the principle was proved in the plane case using the method of exhausting the infinite domain with finite ones. Dealing with our case in the same way we prove the folowing result.

Theorem 4.1. *Let Ω be an admissible domain whose boundary arc Σ was just described.Let u be admissible function so that $(u, \Omega) \in$ D. Let $u_0 = u_0(x, y)$ be a solution of the problem (1.1)–(1.4) in the domainΩ. Then*

$$I_0(u_0, \Omega) \leq I_0(u, \Omega). \tag{4.1}$$

Proof. Let E_h be the class of functions defined in Ω_h with finite integral I_0 assuming in the mean the boundary value equal to unity on the arc Σ and equal to zero on the line $y = -h$. Let us consider the extremal sequence $\nu_n(h)$ for the functional $I_0(u, \Omega_h)$ defined by the condition

$$\lim_{n\to\infty} I_0(\nu_n(h), \Omega_h)$$

$$inf\left\{I_0\left(u,\Omega_h\right),u\in E_h\right\}=d$$

From parallelogram equality we get

$$\frac{I_0\left(\nu_n\left(h\right),\Omega_h\right)+I_0\left(\nu_m\left(h\right),\Omega_h\right)}{2}=I_0\left(\frac{\nu_n\left(h\right)+\nu_m\left(h\right)}{2},\Omega_h\right)+$$
$$+I_0\left(\frac{\nu_n\left(h\right)-\nu_m\left(h\right)}{2},\Omega_h\right)\geq d+I_0\left(\frac{\nu_n\left(h\right)-\nu_m\left(h\right)}{2},\Omega_h\right) \tag{4.2}$$

In accordance with the definition of the constant$d$we get from (4.1) that the sequences $\acute{I_0}\left(\nu_n\left(h\right),\Omega_h\right),\acute{\acute{I_0}}\left(\nu_n\left(h\right),\Omega_h\right)$ are fundamental ones. As for the domains of the considered type the inequality of Friedrichs takes place (see [6]) than the sequence $\nu_n\left(h\right)$is fundamental in the functional space $W^{1,2}\left(\Omega_h\right)$. We denote by $\nu_h=\nu_h\left(x,y\right)$ the limit of this sequence. On the dislocation of the boundary of the domain the functions of the bounded set from the space $W^{1,2}\left(\Omega_h\right)$ behave themselves in equicontinuous way. It is clear that this function satisfies the equation (1.3). Passing to the limit under the sign of the integral we get also that the following condition takes place

$$\lim_{n\to\infty}\int\limits_{-h}^{0}\left(\nu_n\right)^2\left(\pm1,y\right)\,dy=\int\limits_{-h}^{0}\left(\nu_h\right)^2\left(\pm1,y\right)\,dy.$$

Let $h_n:=-n$, and Ω_n,ν_n - sequences corresponding to h_n. Now we are going to construct a solution $\nu=\nu\left(x,y\right)$ of the problem (1.1)–(1.4)considering the sequence $\nu_n$in the domain $\Omega=\bigcup\Omega_n$. Let y_0 be the maximal distance from the points of the curve Σ to the axis x. Let us consider the function $\nu_\infty=\nu_\infty\left(x,y\right)$ defined as follows

$$v_\infty\left(x,y\right)=$$
$$=-\sum_{m=0}^{\infty}\frac{2\alpha}{\lambda_m{}^2\cdot J_0(\lambda_m)\cdot\left[\lambda_m{}^2+\alpha^2\right]}\cdot e^{\lambda_m\cdot(y+y_0)}\cdot\ \ J_0\left(\lambda_m\cdot x\right) \tag{4.3}$$

Here $J_0=J_0\left(t\right)$-Bessel function of zero order and λ_m-solution of the equation

$$\lambda_m\cdot J'{}_0\left(\lambda_m\right)+\alpha\cdot J_0\left(\lambda_m\right)=0$$

In the half-strip Π the function $v_\infty\left(x,y\right)$is the unique solution of the problem (1.1)–(1.4) The difference between the functions $v_\infty\left(x,y\right)-$

$\upsilon_n(x,y) \geq 0, |x| \leq 1, y = y_0$ Taking the condiion (1.3) into account we get that the function$\upsilon_n(x,y)$ satisfies the following condition

$$\nu_n(x,y) \leq \nu_\infty(x,y), (x,y) \in \Pi(y_0) \qquad (4.4)$$

Let us consider the difference $\nu_n - \nu_m, m < n$ in the domain Ω_h. From the condition (4.4)for the points $(x,y) \in (y = -h)$ we get that the limit of this difference for $m$tending to infinity is equal to zero. The difference under consideration is equal to zero on the arc Σ and on the set Γ satisfies to the condition(1.3). It means that the sequence$\nu_n$is fundamental in the sense of the uniform convergence in each domain

$$\overline{\Omega}(n_0) = \overline{\Omega} \bigcap \{y > -n_0\}, n_0 \in N$$

Let $\nu = \nu(x,y)$ be the limit of the sequence $\{\nu_n\}$. It is clear that it satisfies the equation (1.1). The class of the functions admissible for the domain Ω_h does not diminish when $n$increases. It means that the function$\nu = \nu(x,y)$ satisfies the condition (1.2)at the points of the curve Σ. The inequality (4.4) means that this function satisfies also the condition (1.4)at the infinity. It is easy to prove that it also satisfies the condition (1.3)on the set Γ. The above said means that the function $\nu = \nu(x,y)$ satisfies (1.1)–(1.4). It is an admissible function for the the variational problem for the functional $I_0(u,\Omega)$. In the usual way we prove that this functional achieves its minimal value on this function [2]. The theorem is proved.

5. Solution of boundary value problem.

We are going to prove here that using the solutions of the problem (1.1)–(1.4) we are able to construct a solution of the main problem. To begin with we recall that the numberE_0 can be selected arbitrarily large. This permits us to prove the following result.

Lemma 5.1. *For arbitrary non negative values of* λ,κ, θ*there exists a number*E_0*such that the values of the functional* $I(u,\Omega)$ *are non negative on the set of admissible pairs* $(u,\Omega) \in D$.

Proof. Let $Q(u,\Omega)$ be the following expression

$$Q(u,\Omega) := -\lambda \cdot \int\int_B |x| \cdot dx \cdot dy + \theta \cdot \int_\Sigma f(\dot{x}) \cdot ds \qquad (5.1)$$

In accordance with the formula (2.2) we get for the values of E_0 sufficiently large the following inequality

$$Q \geq -\lambda \cdot \theta - \theta \cdot \rho \cdot \int_0^1 \left(\arcsin \sigma + \sigma \cdot \sqrt{1-\sigma^2}\right) \times \\ \times \left(1-\sigma^2\right)^{-\frac{3}{2}} \, d\sigma + \theta \cdot E_0 \cdot \rho \geq 0 \tag{5.2}$$

Here ρ is the maximal distance from the points of the arc $\Sigma$to the axis x. It is clear that the functional $I\,(u,\Omega) - Q\,(u,\Omega)$ assumes nonnegative values over the set of the admissible functions. Now we are ready to prove the main result.

Theorem 5.1. *Let the number $c_0 > 0$, satisfies condition $\kappa - c_0\theta > 0$ Then for all nonnegative α, λ there exists an admissible pair $(u_e, \Omega_e) \in D$, such that*

$$I\,(u_e, \Omega_e) = inf\,\{I\,(u,\Omega)\,, (u,\Omega) \in D\}$$

The curve Σ_e corresponding to the domain Ω is piece-wise analytic and it is nondecreasing for the points with $x > 0$ The function u_e is a solution of the main problem.

Proof. Let$\{e_n\}$ be a minmizing sequence $e_n = (u_n, \Omega_n)$ for the variational problem from the section 2. Without restriction we may assume that the curves $\Sigma_n = \delta\Omega_n \bigcap \Pi$ are piece-wise analytic ones consisting of the regular points(see [7]). The functional Q behaves monotonically under symmetrization of the functions and domains defined before. The functional I behaves itself in the same way. It means that we may assume monotone behaviour of the curves $\Sigma_n$in each quadrant. Using Dirichlet principle we can also assume that the functions u_n are the solutions of the problem (1.1)–(1.4) in the domain Ω_n. In accordance with the lemma 5.1 we can asssume that for the numbers E_0 sufficiently large the functional I is non negative over the set of admissible functions. It means that the set of ordinates of the points of intersection of the curves Σ_n with y - axis is bounded. It follows from the Helly theorem that the sequence of the curves Σ_n converges to the limit curve Σ_e. Besides we get convergence of the domains Ω_n to the domain Ω_e as to the kernel(see [9]). The functions u_n are the solutions of the equation(1.4). It means that we can assume that they are traces of the harmonic functions defined over the domains$\Omega_n^\bullet$ obtained by their rotation about y-axis. From this it follows that the sequence of the functions $u_n$is compact in the sense of the uniform convergence inside of the domain Ω_e. We leave for the convergent subsequence the notation of the proper sequence. Hence the sequence$\{u_n\}$is convergent inside of the domain Ω_e. Let us denote$u_e := \lim\limits_{n\to\infty} u_n(x, y)$. The functions $u_n$are limited by the function ν_∞. It means that the limit function $u_e$assumes at the infinity the value equal to zero. Let us now consider the behaviour of the function u_e on the set $\Sigma_e \bigcup \Gamma_e$. It was said earlier that the functions from the limited set of the Sobolev space are equicontinuous in the mean with regard to the shift of the boundary. The arcs of the set $\Sigma_e - \{x = 0\}$ satisfy Lipshitz condition. It means that the function u_e assumes in the mean the value equal to one on the arc Σ(see[6],[10]). The points of the set $\Sigma_e \backslash \{x = 0\}$ are regular ones. Hence the function u_e assumes on Σ_e the boundary value equal to one with possibl eexeption of the points lying on the y-axis. We can assume that the functions u_n are extended across analytic set Γ. Passing to the limit on Γ we get that the limit function is a solution of equations(1.1)–(1.4). We will show now that the function u_e satisfies almost everywhere on Σ_e the boundary condition (1.5). The functional $I(u, \Omega)$ is semicontinuous from below on the set D (see [4],[7]), whence it follows that it attains its minimal value on the pair $(u_e, \Omega_e) \in D$. Let $z_0 = y_0 + i \cdot x_0$, $0 < x_0 < 1$-be any point on the curve$\Sigma_e$such that tangential line exists at this point. Let $z^\bullet = z + \epsilon \cdot F$ be a transformation defined by the infinitely differentiable function F with support lying in the disk $B(z_0, \delta)$. For the values $\epsilon > 0$ sufficiently small the mapping $z^\bullet(z)$ is the topological one. The Bjorke theorem (see [10]) permits us to consider the function F as the extension on $B(z_0, \delta)$ of the finite infinitely differentiable function given on $\Sigma_e^\delta = \Sigma_e \cap B(z_0, \delta)$. The necessary condition for the element (u_e, Ω_e)

to be extremal can be written in the form

$$0 = \int_0^{|\Sigma(\epsilon)|} \left[\left(2 \cdot i \cdot \lambda \cdot x \cdot \bar{\dot{z}} + 4 \cdot \mathrm{i} \cdot u_{\mathrm{ez}}^2 \mathrm{x} - \frac{\kappa}{i} \right) F \right] ds$$

$$- \int_0^{|\Sigma(\epsilon)|} \left[(\kappa \cdot \mathrm{x} \cdot \bar{\dot{z}} - 2 \cdot \mathrm{i} \cdot \theta \cdot \mathrm{f}_{\dot{\mathrm{x}}} \cdot + \dot{y} \cdot \bar{\dot{z}} + 2 \cdot \theta \cdot f \cdot \bar{\dot{z}}) \frac{dF}{ds} \right] ds \tag{5.3}$$

Here

$$z = y + i \cdot x, u_{ez} = 2^{-1} \cdot \left(\frac{\delta u_e}{\delta y} - i \frac{\delta u_e}{\delta x} \right)$$

and $|\sigma(\epsilon)|$ is the length of the curve $\Sigma_e^\epsilon = \Sigma_e \bigcap \{|z - z_0| < \epsilon\}$. The condition (see(5.2)), is the condition of existence of generalized derivative for the function $\kappa \cdot x \cdot \bar{\dot{z}} - 2 \cdot \mathrm{i} \cdot \theta \cdot \mathrm{f}_{\dot{\mathrm{x}}} \cdot \dot{y} \cdot \bar{\dot{z}} + 2 \cdot \theta \cdot \mathrm{f} \cdot \bar{\dot{\mathrm{z}}} \in \mathrm{L}_2([0, |\Sigma_\epsilon|])$. Let $\Phi_1 = \Phi_1(s), \Phi_2 = \Phi_2(s)$ are the following functions

$$\Phi_1(s) = \kappa \cdot x \cdot \dot{y} - 2\theta \cdot f_{\dot{x}} \cdot \dot{x} \cdot \dot{y} + 2\theta \cdot f \cdot \dot{y}, \tag{5.4}$$

$$\Phi_2(s) = -\kappa \cdot x \cdot \dot{x} - 2\theta \cdot f_{\dot{x}} \cdot \dot{y}^2 - 2\theta \cdot f \cdot \dot{x} \tag{5.5}$$

From above said it follows that they are absolutely continuous. The following equations are the consequences of the conditions

$$G(\dot{y}, \Phi_1, \Phi_2, y) = 0, G = \dot{y} \cdot \Phi_1 - \sqrt{1 - \dot{y}^2} \cdot \Phi_2 - 2 \cdot \theta \cdot f - \kappa \cdot y \tag{5.6}$$

$$H(\dot{x}, \Phi_1, \Phi_2) = 0, H = \dot{x} \cdot \Phi_1 + \sqrt{1 - \dot{\mathrm{x}}^2} \cdot \Phi_2 + 2 \cdot \theta \cdot \mathrm{f}_{\dot{\mathrm{x}}} \cdot \sqrt{1 - \dot{\mathrm{x}}^2}, \tag{5.7}$$

It is easy to show that these equations are solvable in $\dot{x}$, $\dot{y}$ for the values of the parameterE_0 sufficiently large. We get also that $\dot{x} = \Psi_1(\Phi_1, \Phi_2)$, $\dot{y} = \Psi(\Phi_1 \Phi_2)$ for some differentiable functions Ψ_1, Ψ_2. The functions Φ_1, Φ_2, y are absolutely continuous as the functions of the parameter s of natural parametrization of the curve Σ_e. It means that the functions $\dot{x}$, $\dot{y}$ are differentiable almost everywhere on Σ_e^ϵ. Taking into account the above-mentioned results we get from (5.3) the equality

$$-i \cdot \kappa \cdot \dot{z} \cdot x \cdot k(z) + \kappa \cdot \dot{x} \cdot \bar{\dot{z}} + 2 \cdot \imath\, \theta \cdot \ddot{x} \cdot \bar{\dot{z}} = \tag{5.8}$$

$$-2 \cdot i \cdot \lambda \cdot x \cdot \bar{\dot{z}} - 4 \cdot i \cdot \left(\frac{\delta u_e}{\delta z} \right)^2 \cdot \dot{z} + \frac{\kappa}{i} \tag{5.9}$$

Here $k(z)$-curvature of the curve Σ_e^ϵ. In the axially symmetric case we have

$$2 \cdot H(z) = k(z) + \left(\frac{\delta u_e}{\delta z}\right) \cdot (x \cdot |\nabla u_e|)^{-1} \tag{5.10}$$

$$K(z) = -\frac{\ddot{x}}{x}. \tag{5.11}$$

Using (5.6)-(5.8) we get that the function u_e satisfies almost everywhere on the curve Σ_e boundary condition (1.5). Using (5.6)–(5.8) we get that the function u_e satisfies almost everywhere on the curve Σ_e boundary condition (1.5).

Using a‘priory estimates for $|\nabla u_e|$ from the paper [2] we get that the curve $\Sigma_e \bigcap \{x > 0\}$ is a Liapunov curve. From Shauder estimates it follows that this curve is infinitely differentiable one. In the usual way (see [4], [7]) we prove that the curve Σ consists of the analytic arcs.

The theorem is proved.

6. Free boundary.

In this part we investigate the contact of free boundary with the lines $\{x = \pm 1\}$ and $\{y = 0\}$. Let us begin with the set $\left\{\bar{\Sigma} \cap \{x = 1\}\right\}$. We shall prove for the first the following lemma which is the simple generalization of the result proved in ([1]).

Lemma 6.1 *Let b be the length of the segment $\bar{\Sigma} \cap \{x = 1\}$ and $\{u_n\}$-the sequence of the functions from extremal sequence $\{u_n, \Omega_n\}$. Then*

$$\lim_{n\to\infty} \int_{-b}^{0} u_n(1, y)\, dy = b \tag{6.1}$$

Proof. Let us consider the parts $\Sigma_n = \delta\Omega_n \cap \Pi$ lying in the half space $\{y > -b + \epsilon_0\}$. Then from assumption of the lemma it follows that they are contained in the rectangular

$$\Pi_\epsilon = (-b + \epsilon_0 < y < 0\ ) \times (1 - \epsilon, 1)\, , 0 < \epsilon < 1, 0 < \epsilon_0 < b$$

Let us compare the functions u_n, $n > n(\epsilon, \epsilon_0)$, in the rectangular $\Pi_\epsilon$with the function$u_\epsilon$which is a solution of (1.1) and satisfies the following conditions

$$u_\epsilon(x, b-\epsilon) = 0 = u_\epsilon(x, 0), 1-\epsilon < x < 1 \tag{6.2}$$

$$u_\epsilon(1-\epsilon, y) = 1, -b+\epsilon_0 < y < 0 \tag{6.3}$$

$$\frac{du_\varepsilon}{dx} + \alpha \cdot \mathrm{u}_\varepsilon(1, y) = 0, -\mathrm{b}+\varepsilon_0 < y < 0 \tag{6.4}$$

The function $u_\varepsilon$can be represented in the following form

$$u_\varepsilon(x, y) = \sum_{m \geq 0} A_m \cdot V_m(x, \varepsilon) \cdot \mathrm{sinm}\frac{\pi}{-\mathrm{b}+\varepsilon_0} \cdot \mathrm{y} \tag{6.5}$$

$$V_m(x, y) = -\frac{\alpha}{\mathrm{m}} \cdot \mathrm{Y}_m^1(x, \varepsilon) + \mathrm{Y}_{\mathrm{m}}^2(x, \varepsilon) \tag{6.6}$$

$$y'' + \mathrm{x}^{-1} \cdot y' - \mathrm{m}^2 \cdot y = 0 \tag{6.7}$$

$$Y_m^1(1, \varepsilon) = 0, \left(Y_m^1\right)'(1, \varepsilon) = 1 \tag{6.8}$$

$$\mathrm{Y}_m^2(1, \varepsilon) = 1, \left(\mathrm{Y}_m^2\right)'(1, \varepsilon) = 0 \tag{6.9}$$

The functions Y_m^k, $\mathrm{k} = 1, 2$, are constructed as linear combinations of the functions $l_0(mx), \mathrm{K}_0(mx)$, constituting the fundamental system for equation (6.6) (see [11]). It is can be easily verified that

$$\mathrm{Y}_m^1(x) = \frac{-K_0(m) \cdot l_0(m \cdot x) + l_0(m) \cdot K_0(m \cdot x)}{m \cdot l_0(m) \cdot K'_0(m) - l'_0(m) \cdot K_0(m)} \tag{6.10}$$

$$\mathrm{Y}_m^2(x) = \frac{-K_0'(m) \cdot l_0(m \cdot x) - l_0(m) \cdot K_0(m \cdot x)}{l_0(m) \cdot K'_0(m) - l'_0(m) \cdot K_0(m)} \tag{6.11}$$

Using integral representations for the functions $l_0(mx), \mathrm{K}_0(mx)$ (see [11]) we obtain the following estimate for the functions V_m

$$V_m(1-\varepsilon, \varepsilon_0) > \frac{\mathrm{m}}{4} \cdot e^{\varepsilon \cdot \frac{m}{2}} \tag{6.12}$$

The functions u_ε from (6.5) evidently satisfy the condition (6.2). Let us take coefficients A_m in the form

$$A_m = \frac{4}{\pi \cdot m \cdot V_m(1-\varepsilon, \varepsilon_0)},$$

Than we get that the function $u_\varepsilon$satisfies also the condition (6.3). The estimate (6.12) is not exact but it permits us to differentiate the series (6.5)by terms. This means that the function u_ε satisfies the condition (1.1). Thus we have that the functionu_ε is a solution of the problem (1.1),(6.2)–(6.4). Now we evidently have

$$\int_{-b+\epsilon_0}^{0} u_\epsilon(1,y)\,dy = \int_{-b+\epsilon_0^0 \sum_{m=0}}^{m_0} \frac{1}{4\cdot\pi\cdot V_m(1-\epsilon,\epsilon_0)}\cdot\sin\frac{\pi}{-b+\epsilon_0}y\cdot dy+$$

$$+\sum_{m\geq m_0}\frac{4(-b+\epsilon_0)}{\pi\cdot m^2\cdot V_m(1-\epsilon,\epsilon_0)}\cdot\left[1-(-1)^m\right]$$

For a given positive number ϵ_0 we can choose the numbers m_0, $n(\epsilon_0)$ sufficiently large and a positive number ϵ sufficiently small such that for $n > n(\epsilon_0)$ we get the inequality

$$0 \leq -\int_{-b+\epsilon_0}^{0} u_n(1,y)\cdot\,dy + b \leq 2\cdot\epsilon 0$$

Tending n to infinity we get the result we need. The lemma is proved.

Theorem 6.1. *Let (u,Ω) be a solution of variational problem from the point 2. Let us assume that the inequality$\lambda < \alpha$ takes place.Then for the arc $\Sigma_e$of the boundary $\Omega_e$connecting the points $(-1,0),(1,0)$we have*

$$\bar{\Sigma}_e \cap \{x=\pm 1\} = \{(-1,0),(1,0)\} \tag{6.13}$$

Proof. Let us suppose that the theorem is not correct. In this case the number b from the precedent lemma is positive. Using Green theorem we get

$$\int\int_{\Omega_e} |x|\cdot|\nabla u|^2\,dx\,dy + \alpha\cdot\int_\Gamma u_e^2\,dy = \alpha\cdot\int_\Gamma u_e\,dy \tag{6.14}$$

Let us consider the pair $(u^\bullet,\Omega^\bullet)$ for which $u^\bullet(x,y)=u(x,y-b)$and $\Omega^\bullet$is the domain obtained by the dislocation by the number b of the domain Ω in the positive direction of the axis y. The pair $(u^{\bullet}, \Omega^{\bullet})$, evidently, belongs to the set D. We denote as $\Sigma^{\bullet}$ the part of the boundary of $\Omega^{\bullet}$ lying in the closure of the half-strip Π and connecting the points $(-1,0)\,,(1,0)\,.$ Taking (6.13) into account we get

$$I_0\left(u^{\bullet}, \Omega^{\bullet}\right) = I_0\left(u, \Omega\right) + 2 \cdot b \cdot (\lambda - \alpha) - \frac{b^2}{2} \tag{6.15}$$

$$\int\limits_{\Sigma^{\bullet}} |x| \cdot \, ds < \int\limits_{\Sigma} |x| \cdot \, ds \tag{6.16}$$

$$\int\limits_{0}^{|l^{\bullet}|} f\left(\dot{x}\right) \cdot \, ds = \int\limits_{0}^{|l|} f\left(\dot{x}\right) \cdot \, ds \tag{6.17}$$

Here l,l$^{\bullet}$are the lengths of the curves Σ,$\Sigma^{\bullet}$respectively. From the conditions (6.13)–(6.16) it follows that

$$I\left(u^{\bullet}, \Omega^{\bullet}\right) < I\left(u, \Omega\right).$$

The theorem is proved.

We will show now that the curve $\overline{\Sigma}_e = \partial\Omega \bigcap \bar{\Pi}$ for extremal pair (u_e, Ω_e) does not have points on the line $\{y = 0\}$ for the exception of its end points.

Lemma 6.2. *Let* $u^0 = u^0\left(x, y\right)$ *be a solution of the problem (1.1)-(1.4) in the half strip* Π*. Then there exist a number* M *such that for a sufficiently small neighbourhood of the point* $(1,0)$ *the following inequality takes place*

$$u^0_y\left(x, y\right) \geq M \cdot \ln\left[1 + \frac{1}{\alpha \cdot (1 - x)}\right] \tag{6.18}$$

Proof. Without any difficulties we get for the function u^0 the following representation

$$u^0\left(x, y\right) = 1 + \frac{2 \cdot \alpha}{\pi} \cdot \int\limits_{-\infty}^{0} \frac{I_0\left(\mu \cdot x\right) \cdot \sin \mu \cdot \mathrm{y}}{\mu \cdot \left[\mu \cdot \mathrm{I}_1\left(\mu\right) + \alpha \cdot \mathrm{I}_0\left(\mu\right)\right]} rmd\mu \tag{6.19}$$

Here I_0, I_1-Bessel functions of purely imaginary argument. The possibility of such representation follows from asymptotic estimate (see [12])

$$I_n(\mu) = \frac{e^{\mu}}{\sqrt{2\pi \cdot \mu}} \cdot \left[1 + O\left(\mu^{\frac{-1}{2}}\right)\right], \mu \to \infty \tag{6.20}$$

It follows from (6.19) that

$$\begin{aligned} |u_y^0(x,0)| \quad &> \frac{2\alpha}{\pi} \cdot \int_0^{N_1} \frac{I_0(\mu \cdot x)}{[\mu \cdot I_0'(\mu) + \alpha \cdot \mathrm{I}_0(\mu)]} d\mu \\ &+ 3^{-1} \cdot \int_{N_1}^{(1-x)^{-1})} \frac{e^{\mu \cdot (x-1)}}{\mu + \alpha} d\mu \end{aligned} \tag{6.21}$$

The number N_1 in the condition (6.21) is selected in such way that for the function $O\left(\mu^{\frac{-1}{2}}\right)$ from (6.20) takes place inequality

$$\left|O\left(\mu^{\frac{-1}{2}}\right)\right| < \frac{1}{2}.$$

From inequality (6.21) we now get

$$\begin{aligned} &|u_y^0(x,0)| > \frac{2\alpha}{\pi} \cdot \int_0^{N_1} \frac{I_0(\mu x)}{[\mu I_1(\mu) + \alpha I_0(\mu)]} d\mu \\ &+3^{-1}e^{-1} \cdot \ln\left[\frac{1}{1-x} + \alpha\right] - 3^{-1}e^{-1} \cdot \ln\left[\frac{1}{N_1} + \alpha\right] \end{aligned} \tag{6.22}$$

The inequality (6.22) implies the result we need. The lemma is proved

Lemma 6.3. *Let (u, Ω) be a solution of variational problem from section 2 such that $\Sigma \cap (-1, 1) \neq \emptyset$. Then the function u_y tends to infinity logarithmically when x tends to unit.*

Proof. Let us extend the function u as unity to the half strip Π. Let us consider the function $w := u - u^0$ in the half strip Π. We can extend the function w as an odd function across the axis x. From assumption of the lemma it follows that there exists a positive number such that

$\Sigma \cap (0,1) = [\mathrm{c},1)\,, \mathrm{c} < 1$. We consider now the reduction of the extended function w to the set

$$\Pi_c = \{(x,y)\,|0 < c < 1\,, -\infty < y < \infty\}\,.$$

Let us consider sine transform

$$w_s\,(c,\mu) := \int_{-\infty}^{0} w\,(c,y) \cdot \sin\mu \cdot y \mathrm{dy}$$

of the function $w\,(c,y)$. We can easily verify that for the function w in the strip Π_c takes place the following representation.

$$\begin{gathered} w\,(\mathrm{x},\mathrm{y}) = \tfrac{2}{\pi} \cdot \int_{-\infty}^{0} w_s\,(c,\mu)\times \\ \times \tfrac{\mu \cdot I_0[2\mu\cdot(1-x)] + \alpha\cdot I_1[2\mu\cdot(1-x)]}{\mu\cdot I_0[2\mu\cdot(1-c)] + \alpha\cdot I_1[2\mu\cdot(1-c)]} \cdot \sin\mu \cdot y \mathrm{d}\mu \end{gathered} \tag{6.23}$$

From the representation (6.23) it follows that that the function $w_y\,(x,0)$ is bounded in the neighbourhood of the point $(1,0)$. It means that the function $u_y\,(x,o)$ jointly with the function $u_y^0\,(x,0)$ logarithmically tends to the infinity. The lemma is proved.

Theorem 6.2. *Let (u_e,Ω_e) be a solution of variational problem from section 2. Let us suppose that for the numbers λ, α takes place inequality $\lambda^2 < \alpha$. Then the number c, $[\mathrm{c},1) = \Sigma_e \cap (0,1)$, is equal to unit.*

Proof. From the theorem 5.1 it follows that the curve Σ_e consists of the analytic curves. Let us consider variation δI under local variation $\delta\tilde{\mathrm{x}}$ of the boundary

$$\begin{gathered} \delta I\,(\Omega_e,u_e,\delta\vec{x},\delta u) = \int_{\Sigma_e} \left[\lambda - |\nabla u_e|^2\right] \cdot \delta\vec{x} \cdot \vec{\nu} \cdot\, ds + \\ +\kappa \cdot \delta \int_{\Sigma_e} |x| \cdot\, ds + \theta \cdot \delta \int_{\Sigma_e} f\,(\dot{x}) \cdot\, ds \end{gathered} \tag{6.24}$$

We can select as $\delta\tilde{\mathrm{x}}$ the function whose graph for $x > 0$ represents a step with its basis of the length

$$\left|\ln^{\frac{-1}{2}}(1-x_0)\right|$$

centered at the point x_0 and of the same height. In this case we have$h$

$$\delta \int\limits_{\Sigma_e} |x| \ ds + \delta \int\limits_{\Sigma_e} f\left(\dot{x}\right)\ ds = O\left(\left|\ln^{\frac{-1}{2}}\left(1-x_0\right)\right|\right), x_0 \to 1 \qquad (6.25)$$

From lemma 6.3 we get

$$\int\limits_{\Sigma_e} \left[\lambda^2 - \left|\nabla u_e\right|^2\right] \cdot \delta\tilde{\mathrm{x}} \cdot \vec{\nu} \cdot \frac{ds}{\mathrm{x}} + \kappa \cdot \delta \int\limits_{\Sigma_e} |\mathrm{x}| \cdot ds + \theta \cdot \delta \int\limits_{\Sigma_e} \mathrm{f}\left(\dot{\mathrm{x}}\right) \cdot ds < 0 \quad (6.26)$$

It means that

$$\delta \mathrm{I}\left(\Omega_e, \mathrm{u_e}, \delta\tilde{\mathrm{x}}, \delta \mathrm{u}\right) < 0$$

for the points x_0 sufficiently near to one. We denote by the letter $\tilde{\Omega}$ the domain obtained from Ω_e with the help of displacement $\delta\tilde{\mathrm{x}}$. Let $\tilde{u}$ be a solution of boundary problem (1.1)-(1.4) in the domain $\tilde{\Omega}$. On the basis of inequality (6.26) and Dirichlet principle we now get

$$I\left(\tilde{u}, \tilde{\Omega}\right) < I\left(u_e, \Omega_e\right) \qquad (6.27)$$

Let us symmetrize the function $\tilde{u}$ and domain $\tilde{\Omega}$ in order to axis y. We denote as $\tilde{U}$ the symmetrization of the function $\tilde{u}$ and through $\Omega^\bullet$ the symmetrization of the domain $\tilde{\Omega}$ in regard to the axis y. Then

$$\int\limits_{-\infty}^{x_0} U^2\left(1, y\right) \cdot \ dy \le \int\limits_{-\infty}^{0} \tilde{u^2}\left(1, y\right) \cdot \ dy \qquad (6.28)$$

Consider now the function $\acute{U}\left(x, y\right) = U\left(x, y + h\right)$ and domain $\acute{\Omega}$ obtained from domain $\Omega^\bullet$ by dislocation in the negative direction of the axis $y$defined by the number$h$. The pair $\left(\acute{u}, \acute{\Omega}\right)$ is admissible for the extremal problem from the part 2. Using inequalities (6.27)-(6.28)we arrive at the condition

$$I\left(\acute{U}, \acute{\Omega}\right) < I\left(u_e, \Omega_e\right) + 2 \cdot \left(\alpha - \lambda\right) \cdot \ln^{\frac{-1}{2}}\left|\left(1 - x_0\right)\right| \qquad (6.29)$$

The function $\left[\lambda - \left|\nabla u_e\right|^2\right] \cdot \delta\tilde{\mathrm{x}}$ is negative and is of the order $\ln^{-\frac{1}{2}}\left|\left(1 - x_0\right)\right|$ when x_0 tends to 1. The length of the supporter of $\delta\tilde{\mathrm{x}}$

is equal to $\ln^{-\frac{1}{2}}|(1-x_0)|$. It means that for the points x_0 sufficiently near the unity we get on the basis (6.29) the inequality

$$I\left(\acute{U},\acute{\Omega}\right) < I\left(u_e,\Omega_e\right) \tag{6.30}$$

As it was already said the pair $\left(\acute{U}\acute{\Omega}\right)$ is admissible. Thus the inequality (6.30)means that the assumption c <1 leads us to the contradiction. The theorem is proved.

1. *Bazalii B.V. and Shelepov V.Yu.* Variational Methods in a Mixed Problem of Thermal Equilibrium with a Free Boundary // Amer. Math. Soc. Transl., (2)126, (1985), 77-92 .
2. *Gilbarg D. and Trudinger N.S.* Elliptic Partial Differential Equations of Second Order, Springer Verlag, Berlin, Heidelberg, 1983.
3. *Polya G.and Szego G.* Isoperimetric Inequalities in Mathematical Physics, Princeton, Univ.Press,1951.
4. *Garabedian P.R., Lewy H. and Shiffer M.* Axially Symmetric Cavitational Flow // Annals of Mathematics, 56(1952), 560-602.
5. *Garabedian P.R.* Partial Differential Equations, New York-London-Sydney, J.Willey, Inc., 1964.
6. *Michlin S.G.* Linear Partial Differential Equations, Moscow, Vischaya Shkola, 1977(in Russian).
7. *Chthterbakov E.* Free Boundary Value Problem for Axisymmetric Fluid‘s Flow with Surface Tension and Wedging Forces, // Zeitschrift fur Analysis und ihre Anwendungen,17(4), (1998), 937-961.
8. *Kolmogorov A.N. and Fomin S.V.* Elements of the Functional Analysis and Theory of Functions, Moscow, Izdatelstvo Fiziko-matematicheskoi Literatury, 1972 (in Russian).
9. *Suvorov G.D.* Families of Topological Mappings, Novosibirsk,1965 (in Russian).
10. *Kufner J., John O. and Fucik S.* Function Spaces, Prague, Academia, 1977.
11. *Tichonov A.N. and Samarskii A.A.* Equations of Mathematical Physics, Moscow, Nauka, 1966 (in Russian).
12. *Zorich V.A.* Mathematical Analysis, v. 2, Moscow, 2002 (in Russian).

Kuban State University
149 Stavropolskaya st.,
350040, Krasnodar, Russian Federation

Received 1.03.07