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# On conformal invariants in problems of constructive function theory on sets of the real line

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**Abstract.** This is a survey of some recent results by the author and his collaborators in the constructive theory of functions of a real variable. The results are achieved by the application of methods and techniques of modern geometric function theory and potential theory in the complex plane.

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## 1. Introduction

Let $E \subset \mathbb{C}$ be a compact set of positive logarithmic capacity cap$(E)$ with connected complement $\Omega := \overline{\mathbb{C}} \setminus E$ with respect to the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $g_\Omega(z) = g_\Omega(z, \infty)$ be the Green function of $\Omega$ with pole at infinity, and $\mu_E$ be the equilibrium measure for the set $E$ (see [48], [41] for further details on logarithmic potential theory). The properties of $g_\Omega$ and $\mu_E$ play an important role in many problems concerning polynomial approximation of continuous functions on $E$ and the behavior of polynomials with a known uniform norm along $E$.

In this survey we discuss some of these problems for the case when $E$ is a subset of the real line $\mathbb{R}$. The main idea of our approach is to use conformal invariants such as the extremal length and module of a family of curves. The basic conformal mapping can be described as follows.

Let $E \subset [0,1]$ be a regular set such that $0 \in E$, $1 \in E$. Then $[0,1] \setminus E = \sum_{j=1}^{N}(a_j, b_j)$, where $N$ is finite or infinite.

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Denote by $\mathbb{H} := \{z : \Im(z) > 0\}$ the upper half-plane and consider the function

$$F(z) = F_E(z) := \exp\left(\int_E \log(z-\zeta)\,d\mu_E(\zeta) - \log\operatorname{cap}(E)\right), \quad z \in \mathbb{H}. \tag{1.1}$$

It is analytic in $\mathbb{H}$.

Since

$$g_\Omega(z) = \log\frac{1}{\operatorname{cap}(E)} - \int \log\frac{1}{|z-t|}d\mu_E(t), \quad z \in \Omega, \tag{1.2}$$

the function $F$ has the following obvious properties:

$$|F(z)| = e^{g_\Omega(z)} > 1, \quad z \in \mathbb{H},$$

$$\Im(F(z)) = e^{g_\Omega(z)}\sin\left(\int_E \arg(z-\zeta)\,d\mu_E(\zeta)\right) > 0, \quad z \in \mathbb{H}.$$

Moreover, $F$ can be extended from $\mathbb{H}$ continuously to $\overline{\mathbb{H}}$ such that

$$|F(z)| = 1, \quad z \in E,$$

$$F(x) = e^{g_\Omega(x)} > 1, \quad x \in \mathbb{R},\, x > 1,$$

$$F(x) = -e^{g_\Omega(x)} < -1, \quad x \in \mathbb{R},\, x < -1.$$

Next, for any $1 \le j \le N$ and $a_j \le x_1 < x_2 \le b_j$, we have

$$\arg\left(\frac{F(x_2)}{F(x_1)}\right) = \arg\exp\left(\int_E \log\frac{x_2-\zeta}{x_1-\zeta}\,d\mu_E(\zeta)\right) = 0,$$

that is,

$$\arg F(x_1) = \arg F(x_2), \quad a_j \le x_1 < x_2 \le b_j.$$

Our next objective is to show that $F$ is univalent in $\mathbb{H}$. We shall use the following simple result. Let $\sqrt{z^2-1}$, $z \in \overline{\mathbb{C}} \setminus [-1,1]$, be the analytic function defined in a neighborhood of infinity as

$$\sqrt{z^2-1} = z\left(1 - \frac{1}{2z^2} + \dots\right).$$

Then for any $-1 \le x \le 1$ and $z \in \mathbb{H}$,

$$u_x(z) := \Re\left(\frac{\sqrt{z^2-1}}{z-x}\right) \ge 0. \tag{1.3}$$

Using the reflection principle we can extend $F$ to a function analytic in $\overline{\mathbb{C}} \setminus [0,1]$ by the formula

$$F(z) := \overline{F(\overline{z})}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{H}},$$

and consider the function

$$h(w) := \frac{1}{F(J(w))}, \quad w \in \mathbb{D} := \{w : |w| < 1\},$$

where $J$ is a linear transformation of the Joukowski mapping, namely

$$J(w) := \frac{1}{2}\left(\frac{1}{2}\left(w + \frac{1}{w}\right) + 1\right),$$

which maps the unit disk $\mathbb{D}$ onto $\overline{\mathbb{C}} \setminus [0,1]$. Note that the inverse mapping is defined as follows

$$w = J^{-1}(z) = (2z-1) - \sqrt{(2z-1)^2 - 1}, \quad z \in \overline{\mathbb{C}} \setminus [0,1].$$

Therefore, for $z \in \mathbb{H}$ and $w = J^{-1}(z) \in \mathbb{D}$, we obtain

$$\begin{aligned}
\frac{wh'(w)}{h(w)} &= w(\log h(w))' = \\
&= -w\left(\int_E \log(J(w) - \zeta)\, d\mu_E(\zeta)\right)' = -w\, J'(w) \int_E \frac{d\mu_E(\zeta)}{z - \zeta} = \\
&= -\frac{1}{4}\left(w - \frac{1}{w}\right) \int_E \frac{d\mu_E(\zeta)}{z-\zeta} = \frac{1}{2} \int_E \frac{\sqrt{(2z-1)^2 - 1}}{z - \zeta}\, d\mu_E(\zeta) = \\
&= \int_E \frac{\sqrt{(2z-1)^2 - 1}}{(2z-1) - (2\zeta - 1)}\, d\mu_E(\zeta).
\end{aligned}$$

According to (1.3) for $w$ under consideration we obtain

$$\Re\left(\frac{wh'(w)}{h(w)}\right) \geq 0.$$

Because of the symmetry and the maximum principle for harmonic functions we have

$$\Re\left(\frac{wh'(w)}{h(w)}\right) > 0, \quad w \in \mathbb{D}.$$

It means that $h$ is a conformal mapping of $\mathbb{D}$ onto a starlike domain (cf. [35, p. 42]).

Hence, $F$ is univalent and maps $\overline{\mathbb{C}} \setminus [0,1]$ onto a (with respect to $\infty$) starlike domain $\overline{\mathbb{C}} \setminus K_E$ with the following properties: $\overline{\mathbb{C}} \setminus K_E$ is symmetric with respect to the real line $\mathbb{R}$ and coincides with the exterior of the unit disk without $2N$ slits.

Note that

$$g_\Omega(z) = \log|F(z)|, \quad z \in \mathbb{C} \setminus E. \tag{1.4}$$

There is a close connection between the capacities of the compact sets $K_E$ and $E$, namely

$$\mathrm{cap}(E) = \frac{1}{4\,\mathrm{cap}(K_E)}$$

(cf. [5]).

The main idea of the results below is the investigation of the local properties of the Green function $g_\Omega$, i.e., local properties of conformal mapping $F$.

The paper is organized as follows. In section 2 we give a new interpretation (and a generalization) of recent remarkable result by Totik [47] concerning the smoothness properties of $g_\Omega$ and $\mu_E$. We also demonstrate that if for $E \subset [0,1]$ the Green function satisfies the 1/2-Hölder condition locally at the origin, then the density of $E$ at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0,1]$.

In section 3 we describe the connection between $C$-dense compact sets and John domains. It allows us to extend classical Bernshtein's theorem about constructive description of infinitely differentiable functions to the case of $C$-dense compact subset of $\mathbb{R}$.

In section 4 the Nikol'skii-Timan-Dzjadyk theorem concerning polynomial approximation of functions on the interval $[-1,1]$ is generalized to the case of approximation of functions given on a compact set on the real line.

A new necessary condition and a new sufficient condition for the approximation of the reciprocal of an entire function by reciprocals of polynomials on $[0,\infty)$ with geometric speed of convergence are provided in section 5.

In section 6 we give sharp uniform bounds for exponentials of logarithmic potentials if the logarithmic capacity of the subset, where they are at most 1, is known.

We shall use $c, c_1, c_2, \ldots$ to denote positive constants. These constants may be either absolute or they may depend on $E$ depending on the context. We may use the same symbol for different constants if this does not lead to confusion.

## 2. On the Green function for a complement of a finite number of real intervals

The main purpose of this section is to discuss the following recent remarkable result by Totik [47].

Let $E \subset [0,1]$ be a compact set of positive logarithmic capacity. The smoothness of $g_\Omega$ and $\mu_E$ at 0 depends on the density of $E$ at 0. This smoothness can be measured by the function

$$\theta_E(t) := |[0,t] \setminus E|, \quad t > 0,$$

where $|\cdot|$ denotes linear Lebesgue measure.

**Theorem 1.** *(Totik [47, (2.8) and (2.12)]) There are absolute positive constants $C_1, C_2, D_1$ and $D_2$ such that for $0 < r < 1$,*

$$g_\Omega(-r) \le C_1 \sqrt{r} \exp\left( D_1 \int_r^1 \frac{\theta_E^2(t)}{t^3} dt \right) \log \frac{2}{\operatorname{cap}(E)}, \tag{2.1}$$

$$\mu_E([0,r]) \le C_2 \sqrt{r} \exp\left( D_2 \int_r^1 \frac{\theta_E^2(t)}{t^3} dt \right). \tag{2.2}$$

The results in [47] are formulated and proven for general compact sets of the unit disk. The theorem above is one of the main steps in their verification. Even though the statement of this theorem is rather particular, the theorem has several notable applications, such as Phragmén-Lindelöf type theorems, Markov and Bernstein type, Remez and Schur type polynomial inequalities, etc.

Observe that we can simplify geometrical nature of a compact set $E$ under consideration. Indeed, it is well-known that there exists a sequence of compact sets $E_n \subset [0,1]$, $n \in \mathbb{N} := \{1, 2, \ldots\}$, such that

(i) $E \subset E_n$ and each $E_n$ consists of a finite number of closed intervals,

(ii) for $0 < r < 1$, we have

$$g_\Omega(-r) = \lim_{n\to\infty} g_{\Omega_n}(-r), \quad \Omega_n := \overline{\mathbb{C}} \setminus E_n,$$

$$\mu_E([0,r]) = \lim_{n\to\infty} \mu_{E_n}([0,r]).$$

The set $[0,1] \setminus E_n$ is smaller and simpler then $[0,1] \setminus E$. For example,

$$\theta_{E_n}(t) \le \theta_E(t), \quad t > 0.$$

However, $g_{\Omega_n}$ and $\mu_{E_n}$ can be arbitrary close to $g_\Omega$ and $\mu_E$. Thus, in order to establish Totik type results it is natural to concentrate only on compact sets consisting of a finite number of real intervals.

Let

$$E = \cup_{j=1}^{k} [a_j, b_j], \quad 0 \le a_1 < b_1 < a_2 < \ldots < a_k < b_k \le 1,$$

and let

$$E^* := (0,1) \setminus E = \cup_{j=1}^{m} (\alpha_j, \beta_j),\ 0 \le \alpha_1 < \beta_1 < \alpha_2 < \ldots < \alpha_m < \beta_m \le 1.$$

For $0 < r < 1$, we set $E_r^* := E^* \setminus (0, r]$. We are interested in the case when $E_r^* \neq \emptyset$, i.e.,

$$E_r^* = \cup_{j=1}^{m_r} (\alpha_{j,r}, \beta_{j,r}),\ r \le \alpha_{1,r} < \beta_{1,r} < \alpha_{2,r} < \ldots < \alpha_{m_r,r} < \beta_{m_r,r} \le 1.$$

**Theorem 2.** *([4]) For $0 < r < 1$*

$$g_\Omega(-r) \ge c_1 \sqrt{r} \exp \left( d_1 \sum_{j=1}^{m_r} \frac{\beta_{j,r} - \alpha_{j,r}}{\beta_{j,r}} \log \frac{\beta_{j,r}}{\alpha_{j,r}} \right), \tag{2.3}$$

*where $c_1 = 1/16$, $d_1 = 10^{-13}$.*

Theorem 2 provides the lower bound of the Green function (cf. [47, (3.5)]). Since in (2.3) only the size of components of $E_r^*$ influences this bound, one cannot expect to find an upper bound of the same form. We believe that in a Totik type theorem not only the size of the components $(\alpha_{j,r}, \beta_{j,r})$ but also their mutual disposition must be important.

We fix $q > 1$. The set of a finite number of closed intervals

$$\{[\delta_j, \nu_j]\}_{j=1}^{n} = \{[\delta_j(r,q), \nu_j(r,q)]\}_{j=1}^{n},$$

where $0 \le \delta_1 < \nu_1 \le \delta_2 < \ldots \le \delta_n < \nu_n \le 1$, is called a *$q$-covering* of $E_r^*$ if

(i) $E_r^* \subset \cup_{j=1}^{n} [\delta_j, \nu_j]$,
(ii) either $2\delta_j \le \nu_j$, or $q|E_r^* \cap [\delta_j, \nu_j]| \le \nu_j - \delta_j$.

**Theorem 3.** *For $0 < r < 1$, $q > 1$ and any finite $q$-covering of $E_r^*$ the inequalities*

$$g_\Omega(-r) \le c_2 \sqrt{r} \exp \left( d_2 \sum_{j=1}^{n} \frac{\nu_j - \delta_j}{\nu_j} \log \frac{\nu_j}{\delta_j} \right) \log \frac{2}{\operatorname{cap}(E)}, \tag{2.4}$$

$$\mu_E([0,r]) \le c_3 \sqrt{r} \exp \left( d_2 \sum_{j=1}^{n} \frac{\nu_j - \delta_j}{\nu_j} \log \frac{\nu_j}{\delta_j} \right) \tag{2.5}$$

*hold with $c_2 = 24$, $c_3 = 5$ and*

$$d_2 = \max \left( 1, \frac{2q^2}{\pi (q-1)^2} \right).$$

Notice that the factor $\log(2/\mathrm{cap}(E))$ on the right of (2.1) and (2.4) appears only to cover pathological cases. It is useful to keep in mind that $|E| \le 4\,\mathrm{cap}(E) \le 1$.

**Corollary 1.** *The estimates (2.1) and (2.2) hold with $C_1 = 384$, $C_2 = 80$ and $D_1 = D_2 = 120$.*

**Corollary 2.** *For the compact set*

$$\tilde{E} := \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{n^2} \left[ \frac{n^2+j-1}{2^{n+1}n^2}, \frac{2n^2+2j-1}{2^{n+2}n^2} \right].$$

*we have*

$$g_{\overline{\mathbb{C}}\setminus\tilde{E}}(-r) \le c\sqrt{r}, \quad 0 < r < 1,$$

*with some absolute constant $c > 0$, which is better then (2.1).*

Indeed, let

$$\tilde{E}_r := \tilde{E} \cap [r,1], \quad 0 < r < 1.$$

For $\tilde{E}_r^* = (r,1) \setminus \tilde{E}_r$ with $2^{-k-2} < r \le 2^{-k-1}$ we construct a 2-covering

$$[r, 2^{-k}], \left\{ \left\{ \left[ \frac{n^2+j-1}{2^{n+1}n^2}, \frac{n^2+j}{2^{n+1}n^2} \right] \right\}_{j=1}^{n^2} \right\}_{n=1}^{k-1}, \left[ \frac{1}{2}, 1 \right].$$

By the monotonicity of the Green function and Theorem 3, for any $0 < r < 1$ and some absolute constant $c > 0$ we obtain

$$g_{\overline{\mathbb{C}}\setminus\tilde{E}}(-r) \le g_{\overline{\mathbb{C}}\setminus\tilde{E}_r}(-r) \le c\sqrt{r}.$$

In what follows in this section we assume that 0 is a regular point of $E$, i.e., $g_\Omega(z)$ extends continuously to 0 and $g_\Omega(0) = 0$.

The monotonicity of the Green function yields

$$g_\Omega(z) \ge g_{\overline{\mathbb{C}}\setminus[0,1]}(z), \quad z \in \mathbb{C} \setminus [0,1],$$

that is, if $E$ has the "highest density" at 0, then $g_\Omega$ has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r) \ge g_{\overline{\mathbb{C}}\setminus[0,1]}(-r) > \frac{\sqrt{r}}{2}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of $E$ whose Green's function has the “highest smoothness" at 0, that is, of $E$ conforming to the following condition

$$g_\Omega(z) \le c|z|^{1/2}, \quad c = \mathrm{const} > 0,\ z \in \mathbb{C},$$

which is known to be the same as

$$\limsup_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2}} < \infty \tag{2.6}$$

(cf. [41, Corollary III.1.10]). Various sufficient conditions for (2.6) in terms of metric properties of $E$ are stated in [47], where the reader can also find further references.

There are compact sets $E \subset [0,1]$ of linear Lebesgue measure 0 with property (2.6) (see e.g. [47, Corollary 5.2]), hence (2.6) may hold, though the set $E$ is not dense at 0 in terms of linear measure. On the contrary, our first result states that if $E$ satisfies (2.6) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrary close to the density of $[0,1]$ in that neighborhood.

**Theorem 4.** *([5]) The condition (2.6) implies*

$$\lim_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = \frac{1}{4}. \tag{2.7}$$

Recall that $\operatorname{cap}([0,r]) = r/4$ for any $r > 0$.

The converse of Theorem 4 is slightly weaker.

**Theorem 5.** *([5]) If $E$ satisfies* (2.7)*, then*

$$\lim_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0<\varepsilon<\frac{1}{2}\,. \tag{2.8}$$

The connection between properties (2.6), (2.7) and (2.8) is quite delicate. For example, even a slight alteration of (2.6) can lead to the violation of (2.7). As an illustration of this phenomenon we formulate

**Theorem 6.** *([5]) There exists a regular compact set $E \subset [0,1]$ such that* (2.8) *holds and*

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = 0. \tag{2.9}$$

## 3. Constructive description of infinitely differentiable functions on a compact set of the real line

Let $E \subset \mathbb{R}$ be a compact set. Denote $C(E)$ the set of all real functions $f$ continuous on $E$. Let $\mathbf{\Pi}_n, n \in \mathbb{N}$, be the set of all polynomials with real coefficients of degree at most $n$. For $f \in C(E)$ and $n \in \mathbb{N}$ let

$$E_n(f,E) := \inf_{p\in\mathbf{\Pi}_n} ||f-p||_E,$$

where $||\cdot||_E$ is the uniform norm over $E$.

We start with the following classical theorem.

**Theorem 7.** *(Bernstein [10]) The function* $f \in C([a,b])$ *is infinitely differentiable on* $[a,b]$ *iff*

$$\lim_{n\to\infty} E_n(f,[a,b])\, n^c = 0, \quad c > 0. \tag{3.1}$$

Our purpose is to describe the result analogous to Bernstein's theorem for functions given on a general compact set $E$ instead of the interval.

The condition

$$\lim_{n\to\infty} E_n(f,E)\, n^c = 0, \quad c > 0. \tag{3.2}$$

is obvious analogue of (3.1).

We have to pay more attention to the generalization of the notion of differentiability on a subset of $\mathbb{R}$. We follow the approach due to H. Whitney [49] involving the Taylor formula (in finite form). That is, we say that $f =: f_0 \in C(E)$ is of class $C^\infty(E)$ if there exist functions $\{f_j(x)\}_{j=1}^\infty$, $\{R_{sm}(x',x)\}_{s\le m=0}^\infty$, where $x, x' \in E$, such that

$$f_s(x') = \sum_{j=s}^{m} \frac{f_j(x)}{(j-s)!}(x'-x)^{j-s} + R_{sm}(x',x)$$

and for each $s, m$ and every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, s, m) > 0$ such that the inequality

$$\left| \frac{R_{sm}(x',x)}{(x'-x)^{m-s}} \right| < \varepsilon$$

holds for any $x', x \in E$ satisfying $|x'-x| < \delta$.

A description of $f \in C^\infty(E)$ in terms of the divided differences of $f$ with respect to points of $E$ can be found in [50]. A classical Whitney theorem [49, Theorem 1] asserts that $f \in C^\infty(E)$ iff $f$ is a restriction on $E$ of some function infinitely differentiable on $\mathbb{R}$. Hence, we have

**Corollary 3.** *Let* $E \subset \mathbb{R}$ *be an arbitrary compact set. For any* $f \in C^\infty(E)$ *the condition (3.2) holds.*

This statement cannot be converted for an arbitrary set $E$ (see [33], [46]). However, the following result is valid.

**Theorem 8.** *(Pleśniak [34]) Let* $E$ *be such that the Green function* $g_\Omega$ *satisfies the Hölder condition, i.e., there are constants* $c > 0$ *and* $0 < \alpha \le 1/2$ *such that*

$$g_\Omega(z) \le c\, \mathrm{dist}(z,E)^\alpha, \quad z \in \Omega. \tag{3.3}$$

*If* $f \in C(E)$ *satisfies (3.2), then* $f \in C^\infty(E)$.

Thus, Theorem 8 and Corollary 3 extend Bernstein's theorem to compact sets of $\mathbb{R}$ satisfying (3.3). It is natural to study metric properties of such sets. It turns out that $C$-dense compact subsets of $\mathbb{R}$ known in approximation theory of a complex variable satisfy (3.3). This can be derived from the description of the connection between $C$-dense sets and John domains well-known in geometrical function theory and the theory of quasiconformal mappings.

Following Tamrazov [44, p. 61] we call $E$ *$C$-dense* if

$$\liminf_{t\to 0^+} \inf_{x\in E} \frac{\operatorname{cap}(E\cap[x-t,x+t])}{t} > 0.$$

$C$-dense sets arise in many areas of complex analysis under the name *uniformly perfect* sets. This class of sets was originally considered by Beardon and Pommerenke [9]. Almost at the same time Pommerenke [36] proved the equivalence of these notions (for general unbounded closed sets in $\mathbb{C}$).

Let $E\subset[0,1]$, $0\in E$, $1\in E$, and let $F=F_E$ be a conformal mapping defined in section 1, i.e., $F$ map $\overline{\mathbb{C}}\setminus[0,1]$ onto $\overline{\mathbb{C}}\setminus K_E =: \Delta_E$. The domain $\Delta_E$ is called a *John domain* (see [35, p. 96]) if, for every rectilinear crosscut $[z,\zeta]$ of $\Delta_E$,

$$\operatorname{diam}(H)\le c\,|z-\zeta|$$

holds with some constant $c>0$ for the bounded component $H$ of $\Delta_E\setminus[z,\zeta]$.

**Theorem 9.** *([6]) $E$ is a $C$-dense compact set iff $\Delta_E$ is a John domain.*

This theorem builds a bridge between two quite different concepts of analysis and can be used to study properties of $C$-dense sets. In particular, from general properties of John domains and (1.4) we obtain the following.

**Corollary 4.** *For any $C$-dense compact set $E$ the Green function $g_\Omega$ satisfies (3.3).*

Hence, infinitely differentiable functions given on a $C$-dense compact sets can be characterized by the condition (3.2).

## 4. The Nikol'skii-Timan-Dzjadyk-type theorem

Let $E\subset\mathbb{R}$ be a compact set, and let $\omega(\delta),\delta>0$, be a function of modulus of continuity type, i.e., a positive nondecreasing function with $\omega(0+)=0$ such that for some constant $c\ge 1$,

$$\omega(t\delta)\le c\,t\,\omega(\delta),\quad \delta>0, t>1.$$

Let $C_\omega(E)$ consist of all $f \in C(E)$ such that

$$|f(x_1) - f(x_2)| \le c_1\, \omega(|x_2 - x_1|), \quad x_1, x_2 \in E,$$

with some $c_1 = c_1(f) > 0$.

For $\omega(\delta) = \delta^\alpha, 0 < \alpha \le 1$, we set $C_\omega(E) =: C^\alpha(E)$.

One of the central problems in approximation theory is to describe the relation between the smoothness of functions and the rate of decrease of their approximation by polynomials when the degree of these polynomials tends to infinity. The following well-known statement is the starting point of our consideration.

**Theorem 10 (Nikol'skii [32], Timan [45], Dzjadyk [17]).** *Let $f \in C([-1,1])$ and let $\omega$ be a function of modulus of continuity type satisfying the inequality*

$$\delta \int_\delta^1 \frac{\omega(t)}{t^2}\, dt \le c_2\, \omega(\delta), \quad 0 < \delta < 1, \tag{4.1}$$

*with some constant $c_2 > 0$. Then the following assertions are equivalent:*

*(i) $f \in C_\omega([-1,1])$;*

*(ii) for any $n \in \mathbb{N}$ there exists $p_n \in \mathbf{\Pi}_n$ such that the inequality*

$$|f(x) - p_n(x)| \le c_3\, \omega\left(\frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}\right), \quad -1 \le x \le 1, \tag{4.2}$$

*holds with some constant $c_3 > 0$.*

In the late 50s - early 60s Dzyadyk [18], [19] laid the foundation of a new constructive theory of functions on continua in the complex plane (a survey of the results and a bibliography can be found in the monographs [20], [44], [24], [42], [7]). He used the following simple but fundamental idea.

Denote by $I_{1/n}, n \in \mathbb{N}$, the ellipse with foci at $\pm 1$ and sum of semiaxes equal to $1 + 1/n$. Such an ellipse is the image of the circle $\{w : |w| = 1 + 1/n\}$ under the conformal mapping $z = \frac{1}{2}(w + 1/w)$ of $\Delta := \{w : |w| > 1\}$ onto $\overline{\mathbb{C}} \setminus [-1,1]$, i.e., $I_{1/n}$ is the level line of the conformal mapping

$$\Phi(z) = z + \sqrt{z^2 - 1}$$

of $\overline{\mathbb{C}} \setminus [-1,1]$ onto $\Delta$, where the square root is chosen so that $\Phi(z) = 2z + O(\frac{1}{|z|})$ in a neighborhood of $\infty$.

Then for $-1 \le x \le 1$ and $n \in \mathbb{N}$,

$$\frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n} \asymp \rho_{1/n}(x),$$

where $a \asymp b$ means a double inequality

$$\frac{a}{c} \le b \le c\, a$$

with some constant $c \ge 1$, and

$$\rho_{1/n}(x) := \operatorname{dist}(x, I_{1/n}),$$

where

$$\operatorname{dist}(A,B) := \inf_{z\in A, \zeta\in B} |z-\zeta|, \quad A,B \subset \mathbb{C}.$$

The concepts of $C_\omega, \Phi, I_{1/n}$ and $\rho_{1/n}(x)$ are also meaningful for an arbitrary bounded continuum in the complex plane. This is the key to a generalization of the Nikol'skii-Timan-Dzjadyk theorem to classes of functions on continua in $\mathbb{C}$.

If $E \subset \mathbb{C}$ is a compact set, then the interpretation of the Nikol'skii-Timan-Dzjadyk theorem above can be corrected by consideration of the Green function $g_\Omega$ and its level lines. If $E$ consists of a finite number of components, Nguen Tu Than' [31] has found a simple way to reduce the problem to the case of a continuum. The case of infinitely connected $\Omega$ is extremely difficult to handle. This can be seen from the recent paper of Shirokov [43] (the first work devoted to this quite new situation).

We are going to discuss the case $E \subset \mathbb{R}$, where the number of components of $E$ can be infinite. It turns out that the appropriate analogue of the Nikol'skii-Timan-Dzjadyk theorem is valid for some $E$ that are not "too scarce" (see Theorem 12) and that in general a result of such kind is not true (see Theorem 11).

More precisely, let $E \subset \mathbb{R}$ be a regular compact set. For $\delta > 0$ and $z \in \mathbb{C}$ set

$$E_\delta := \{z \in \Omega\, :\, g_\Omega(z) = \delta\},$$

$$\rho_\delta(z) := \operatorname{dist}(z, E_\delta).$$

It turns out that even for $f \in C^\alpha(E)$, polynomials satisfying an analogue of (4.2) cannot be constructed for any $E$ under consideration.

**Theorem 11.** *([1]) There exist a regular compact set $E_0 \subset \mathbb{R}$ and for any $0 < \alpha \le 1$ a function $f_\alpha \in C^\alpha(E_0)$ such that the following assertion is false: for any $n \in \mathbb{N}$ there is a polynomial $p_n \in \mathbf{\Pi}_n$ with the property:*

$$|f_\alpha(x) - p_n(x)| \le c\, \rho^\alpha_{1/n}(x), \quad x \in E_0, \tag{4.3}$$

*where the constant $c > 0$ is independent of $n$ and $x$.*

The analysis of the proof of Theorem 11 shows that $E_0$ is "too scarce" in a neighbourhood of $0 \in E_0$. Hence, to admit estimates like (4.2) or (4.3), $E$ has to be "thick enough" in a neighbourhood of each of its points. In order to formulate the appropriate restrictions we need some notations.

The set $\mathbb{R} \setminus E$ consists of a finite or infinite number of components, i.e., disjoint open intervals.

We say that $E \in \mathcal{E}(\alpha, c)$, $\alpha > 0, c > 0$, if for any bounded component $J$ of $\mathbb{R} \setminus E$ the inequality

$$\operatorname{dist}(J, (\mathbb{R} \setminus E) \setminus J) \geq c\, |J|^{1/(1+\alpha)}$$

holds.

By definition, we relate a single closed interval to $\mathcal{E}(\alpha, c)$.

We can now state the analogue of the Nikol'skii-Timan-Dzjadyk theorem for functions continuous on a compact subset of the real line.

**Theorem 12.** *([1]) Let the regular set $E \subset \mathbb{R}$ consist of a finite number of disjoint compact sets, each of which belongs to the class $\mathcal{E}(\alpha, c)$ with some $\alpha, c > 0$. Suppose that $f \in C(E)$ and that the function $\omega$ of the modulus of continuity type satisfies (4.1).*

*Then the following conditions are equivalent:*

*(i) $f \in C_\omega(E)$;*

*(ii) for any $n \in \mathbb{N}$ there exists a polynomial $p_n \in \mathbf{\Pi}_n$ such that*

$$|f(x) - p_n(x)| \leq c_1\, \omega(\rho_{1/n}(x)), \quad x \in E,$$

*where the constant $c_1 > 0$ does not depend on $x$ and $n$.*

The simplest example of $E$ satisfying the assumptions of Theorem 12 is the union of a finite number of disjoint closed intervals. The compact set

$$E_\alpha := \{0\} \cup \bigcup_{n=n_\alpha}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n} - \frac{1}{n^{2+\alpha}}\right], \quad \alpha > 0,\, n_\alpha > 2^{1/\alpha},$$

which obviously satisfies the conditions of Theorem 12, illustrates a non-trivial extension of (4.2) to compact subsets of the real line.

If $E$ consists of an infinite number of components, then taking $\pm 1$ as values of $f$ on different subintervals of $E$ (that is, $f' \equiv 0$ on $E$) we can construct a function $f$ which is not even continuous on $E$. It shows that unlike to the case of an interval, Theorem 12 does not in general admit a generalization to the classes of functions continuously differentiable on $E$ with given majorant for the modulus of continuity of their derivatives.

## 5. Approximation on an unbounded interval

Let $\mathbb{R}^+$ denote the non–negative real axis. We will consider functions $f$, continuous in the complex plane $\mathbb{C}$, real valued on $\mathbb{R}^+$ and possessing also the basic properties

$$f > 0 \text{ on } \mathbb{R}^+, \quad \lim_{x\to\infty} f(x) = \infty \,. \tag{5.1}$$

For every positive integer $n \in \mathbb{N}$, we define $\rho_n(f)$ as follows:

$$\rho_n(f) := \inf_{p\in\mathbf{\Pi}_n} ||1/f - 1/p||_{\mathbb{R}^+} \,. \tag{5.2}$$

In the present section, we discuss necessary and sufficient conditions for the geometric convergence of reciprocals of polynomials to the reciprocal of the function $f$ on the half–axis $\mathbb{R}^+$, i.e., we shall discuss the inequality

$$\limsup_{n\to\infty} \rho_n(f)^{1/n} = \frac{1}{q} < 1 \,. \tag{5.3}$$

The first results in this area were due to Cody, Meinardus and Varga [15] concerning the function $\exp(x)$. Later, Meinardus and Varga [28] extended these results to the class of entire functions of completely regular growth. The paper [30] gave rise to investigations devoted to enlarging the class of functions that admit geometric approximation by reciprocals of polynomials on $\mathbb{R}^+$.

We introduce some notations. Given numbers $r > 0$ and $s > 1$, denote by $\mathcal{E}_r(s)$ the closed ellipse with foci at the points $x = 0$ and $x = r$ such that the ratio between the semimajor axis and semiminor axis equals $(s^2+1)/(s^2-1)$. Further, let $\mu(r) := \min_{x\geq r}\{f(x)\}$.

The remarkable result concerning necessary conditions for geometric convergence is the following

**Theorem 13.** *(Meinardus [29], Meinardus, Reddy, Taylor, Varga [30]) Let $f$ satisfy (5.3). Then*

*(i) the function $f$ can be extended from $\mathbb{R}^+$ to an entire function of finite order,*

*(ii) for every number $s > 1$, there exist positive constants $c_1 = c_1(s,q)$, $\theta = \theta(s,q)$ and $r_0 = r_0(s,q)$ such that the inequality*

$$||f||_{\mathcal{E}_r(s)} \leq c_1 ||f||^{\theta}_{[0,r]} \tag{5.4}$$

*holds for all $r \geq r_0$.*

After the appearance of [30], a lot of work was done to find sufficient conditions for (5.3) (cf. [11]–[13], [27], [38], [40]). The most general known result in this direction is the following

**Theorem 14.** *(Blatt, Kovacheva [13]) Assume that $f$ is an entire function with (5.1) and, in addition to condition (5.4), the inequality*

$$||f||_{[0,r]} \leq \mu(r)^{\lambda} \tag{5.5}$$

*holds for some number $\lambda > 1$ and for every $r > r_0$. Then (5.3) is true.*

On the other hand, Henry and Roulier [27] have shown that the conditions (i) and (ii) of Theorem 13 are not sufficient for geometric convergence. For example, in [27] it was proved that

$$f(x) = 1 + x + e^{x} \sin^2 x \tag{5.6}$$

cannot be approximated with geometric speed. Their proof was based on the fact that $f$ satisfying (5.3) cannot oscillate too often.

The main aim of this section is to discuss a new necessary and a new sufficient condition for geometric convergence. In this context it is important to note that up to now all proofs for geometric convergence are based on the classical Bernstein theorem [10], i.e.,

$$E_n(f, [0,1]) \leq c\, ||f||_{\mathcal{E}_r(s)}\, s^{-n}\,, \tag{5.7}$$

where the function $f$ is analytic in $\mathcal{E}_r(s)$ and the constant $c$ is independent of the interval $[0, r]$. In our reasoning, we use a new generalization of (5.7) for a finite number of intervals, new in the sense that the constant $c$ is independent of the geometry (Theorem 19).

We begin with a necessary condition for geometric convergence of best approximants (5.2). Let $f$ be as above, i.e., $f$ is an entire function with (5.1). For $r > 0$, we define the set

$$Z_r := \{\, 0 \leq x < \infty : f(x) < r \,\}\,.$$

Then $Z_r$ is the union of a finite number of disjoint open intervals. This follows from (5.1) and the uniqueness theorem for analytic functions. Now, we consider the closure $\bar{Z}_r$ of $Z_r$, which is regular and possesses a Green's function $g_r(z)$ with respect to the region $\overline{\mathbb{C}} \setminus \bar{Z}_r$ with pole at infinity, where $g_r := 0$ on $\bar{Z}_r$. For $s > 1$ we denote by $\mathcal{E}_r(f, s)$ the set which consists of the interior of the level set of $g_r(z)$ and the level set itself for a fixed parameter $s$, i.e.,

$$\mathcal{E}_r(f, s) := \{\, z \in \mathbb{C} \,:\, 0 \leq g_r(z) \leq \log s \,\}\,.$$

Then the new necessary condition for geometric convergence can be formulated as follows.

**Theorem 15.** *([8]) Let $f$ satisfy (5.3). Then for every $1 < s < q$ there exist positive constants $c = c(s,q)$, $\theta = \theta(s,q)$ and $r_0 = r_0(s,q)$ such that*

$$||f||_{\mathcal{E}_r(f,s)} \leq c\, r^{\theta}, \quad r \geq r_0\,. \tag{5.8}$$

Next, we are going to discuss the geometrical meaning of condition (5.8). For $\infty > H > h > \min_{x\in\mathbb{R}^+} f(x) > 0$, we introduce the strip domain

$$S(h,H) := \{\, (x,y) \,:\, -\infty < x < \infty,\ h < y < H \,\}$$

as well as the intersection of this strip with the graph of $f$, i.e.,

$$Y(f,h,H) := S(h,H) \cap \{\, (x,y) \,:\, x \geq 0,\ y = f(x) \,\}\,,$$

and define $N(f,h,H)$ to be the number of connected components of $Y(f,h,H)$ joining the line $\{\Im z = h\}$ with the line $\{\Im z = H\}$. Since $f$ satisfies (5.1), the number $N(f,h,H)$ is finite and, moreover, it is odd.

**Theorem 16.** *([8]) Let $f$ be entire and satisfy (5.1). If, in addition, for some $s > 1$ and $\theta > 1$, the function $f$ satisfies (5.8), then, for each $M > \theta$,*

$$\limsup_{h\to\infty} \frac{N(f,h,h^M)}{\log h} < \infty\,. \tag{5.9}$$

Note that the result of Theorem 16 is sharp in the following sense: For each $M > 1$ there exists an entire function $f = f_M$ which satisfies (5.8) with some $s > 1$ and $1 < \theta < M$ and

$$\limsup_{h\to\infty} \frac{N(f,h,h^M)}{\log h} > 0\,. \tag{5.10}$$

Indeed, consider the function

$$f_M(x) := e^x + e^{2Mx}\sin^2 \pi x\,.$$

Obviously, it satisfies the conditions of Theorem 14. Therefore, $f_M$ guarantees the geometrical convergence of best approximants in the sense of (5.3), and, by Theorem 15, $f$ satisfies (5.8) in which we can take $s$ so close to 1 that $\theta < M$. The relation (5.10) easily follows, if we set $h = e^k$, $k \in \mathbb{N}$, and let $k \to \infty$.

The sufficient condition for geometrical convergence of best approximants can be stated in the following form.

**Theorem 17.** *([8]) Let $f$ be entire and satisfy (5.1) and (5.8) with some $s > 1$ and $\theta > 1$. In addition, assume that there exists a constant $M = M(f) > 1$ such that*

$$\limsup_{h\to\infty} N(f, h, h^M) < \infty\,. \tag{5.11}$$

*Then $f$ satisfies (5.3).*

It is easy to see that Theorem 14 follows from Theorem 17, because under the assumptions of Theorem 14, for $M > \lambda$ and $h$ sufficiently large, we have $N(f, h, h^M) = 1$. At the same time, condition (5.4) is weaker than (5.8). The simple example of the function (5.6) shows that conditions (5.4) and (5.8) are not equivalent. Indeed, $f$ given by (5.6) obviously satisfies (5.4). On the other hand, some simple calculations show that relation (5.9) does not hold for this function. Thus, $f$ does not possess (5.8).

The fact that Theorem 17 is essentially stronger than Theorem 14 is not so obvious.

**Theorem 18.** *([8]) There exists an entire function $f$ satisfying the assumptions of Theorem 17, but not possessing property (5.5).*

The proof of Theorem 17 is based on an analogue of the classical result due to Bernstein, concerning polynomial approximation of functions analytic in the neighborhood of a subinterval of the real axis, for the case of several intervals.

Let $E = \bigcup_{j=1}^{k} I_j$ be the union of $k$ disjoint intervals $I_j = [\alpha_j, \beta_j]$ of the real axis $\mathbb{R}$ and let $\Omega := \overline{\mathbb{C}}\setminus E$. The set

$$E^s := \{z \in \Omega \,:\, g_\Omega(z) = \log s\,\}, \quad s > 1,$$

consists of at most $k$ (mutually exterior) curves. Denote by $\mathrm{ext}(E^s)$ the unbounded component of $\overline{\mathbb{C}}\setminus E^s$ and set $\mathrm{int}(E^s) := \overline{\mathbb{C}}\setminus\overline{\mathrm{ext}(E^s)}$. Moreover, let the function $f \in C(E)$ satisfy the following two conditions:

$$\text{For some } s > 1,\ f \text{ can be extended analytically into } \overline{\mathrm{int}(E^s)}\,, \tag{5.12}$$

$$f \ \text{ has at least one zero on each } \ I_j\,. \tag{5.13}$$

**Theorem 19.** *([8]) For each function $f$ satisfying (5.12) and (5.13), there exist constants $q > 1$ and $c > 0$ depending only on $s$ and $k$ such that*

$$E_n(f, E) \le c\,||f||_{E^s}\, q^{-n}, \quad n \in \mathbb{N}\,. \tag{5.14}$$

Note that (5.14) can be interpreted as a result concerning geometric convergence of the polynomials of best approximation to the function $f$, independent of the geometry of $E$.

## 6. Remez-type inequalities in terms of capacity

The Remez inequality [39] (see also [22, 14, 25]) asserts that

$$||p_n||_I \le T_n\left(\frac{2+s}{2-s}\right) \tag{6.1}$$

for every polynomial $p_n \in \mathbf{\Pi}_n$ such that

$$|\{x \in I : |p_n(x)| \le 1\}| \ge 2-s, \quad 0 < s < 2, \tag{6.2}$$

where $I := [-1,1]$ and $T_n$ is the Chebyshev polynomial of degree $n$.

Since

$$T_n(x) \le (x + \sqrt{x^2-1})^n, \quad x > 1,$$

we have by (6.1) that a polynomial $p_n$ with (6.2) satisfies

$$||p_n||_I \le \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^n. \tag{6.3}$$

It is easy to see that the last inequality (more precisely its $n$-th root) is asymptotically sharp.

Our aim is to discuss an analogue of (6.2)-(6.3) in which we use logarithmic capacity instead of the length. Our main result deals not only with polynomials, but also with exponentials of potentials (see [22, 23]).

We refer to the basic notions of potential theory (such as capacity, potential, Green's function, equilibrium measure, Fekete polynomials, etc.) without special citations. All these notions and their properties can be found in [48, 35, 41].

Given a nonnegative Borel measure $\nu$ with compact support (in the complex plane $\mathbb{C}$) and finite total mass $\nu(\mathbb{C}) > 0$ as well as a constant $c \in \mathbb{R}$, we say that

$$Q_{\nu,c}(z) := \exp(c - U^\nu(z)), \quad z \in \mathbb{C},$$

where

$$U^\nu(z) := \int \log \frac{1}{|\zeta - z|}\, d\nu(\zeta), \quad z \in \mathbb{C},$$

is the logarithmic potential of $\nu$, is an *exponential of a potential* of degree $\nu(\mathbb{C})$.

Let

$$E_{\nu,c} := \{x \in I : Q_{\nu,c}(x) \le 1\}.$$

Theorem 2.1 and Corollary 2.11 in [23] assert that for $0 < s < 2$ the condition

$$|E_{\nu,c}| \ge 2-s \tag{6.4}$$

implies

$$||Q_{\nu,c}||_I \leq \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^{\nu(\mathbb{C})}. \tag{6.5}$$

Our result can be formulated as follows.

**Theorem 20.** *([2] Let* $0<\delta<1/2$. *Then the condition*

$$\operatorname{cap}(E_{\nu,c}) \geq \frac{1}{2}-\delta \tag{6.6}$$

*yields that*

$$||Q_{\nu,c}||_I \leq \left(\frac{1+\sqrt{2\delta}}{1-\sqrt{2\delta}}\right)^{\nu(\mathbb{C})}. \tag{6.7}$$

**Remark 1** Since $|E_{\nu,c}| \leq 4\operatorname{cap}(E_{\nu,c})$ [35, p. 337], the assertion (6.4)-(6.5) follows from (6.6)-(6.7).

**Remark 2** For $0<\delta<1/2$, set

$$\nu=\nu_\delta := \mu_{[-1,1-4\delta]}, \quad c=c_\delta := \log\frac{2}{1-2\delta},$$

where $\mu_E$ denotes the equilibrium measure of a compact set $E\subset\mathbb{C}$.

Then $\nu(\mathbb{C})=1$,

$$E_{\nu,c}=[-1,1-4\delta],$$

$$Q_{\nu,c}(x)=\frac{1}{1-2\delta}\left(x+2\delta+((x+2\delta)^2-(1-2\delta)^2)^{1/2}\right), \quad x\geq 1-4\delta.$$

Therefore, in this case

$$\operatorname{cap}(E_{\nu,c})=\frac{1}{2}-\delta,$$

$$||Q_{\nu,c}||_I = Q_{\nu,c}(1)=\frac{1+\sqrt{2\delta}}{1-\sqrt{2\delta}},$$

which shows the sharpness of Theorem 20.

**Remark 3** Let $p_n(z)=c\prod_{j=1}^n(z-z_j)$, $0\neq c\in\mathbb{C}$, be a complex polynomial of degree $n$, and let

$$\nu_n := \sum_{j=1}^n \delta_{z_j},$$

where $\delta_z$ is the Dirac unit measure in a point $z\in\mathbb{C}$. For $z\in\mathbb{C}$, we have

$$Q_{\nu_n,\log|c|} = \exp\left(\log|c|+\log\prod_{j=1}^n|z-z_j|\right)=|p_n(z)|.$$

Therefore, applying the theorem we obtain for $0 < \delta < 1/2$: the condition

$$\operatorname{cap}(\{x \in I : |p_n(x)| \leq 1\}) \geq \frac{1}{2} - \delta$$

implies

$$||p_n||_I \leq \left( \frac{1 + \sqrt{2\delta}}{1 - \sqrt{2\delta}} \right)^n$$

(cf. (6.2)-(6.3)).

**Remark 4** The previous remark can be rewritten in a form as in [16, Theorem 1.1]. Namely, let $r > 0$ and $p_n$ be a complex polynomial of degree $n$ such that $||p_n||_{[-r,r]} = 1$. Then for $0 < \varepsilon < 1$,

$$\operatorname{cap}(\{x \in [-r, r] : |p_n(x)| \leq \varepsilon^n\}) \leq \frac{2r\varepsilon}{(1 + \varepsilon)^2}.$$

This inequality is asymptotically sharp for any fixed $\varepsilon$ and $r$.

Note that Theorem 20 is a straightforward consequence of its following particular case.

**Lemma 1.** *([2]) Let* $E \subset I$ *be a compact set with* $0 < \operatorname{cap}(E) < 1/2$. *Then*

$$\sup_{x \in I \setminus E} g_{\overline{\mathbb{C}} \setminus E}(x) \leq \log \left( \frac{1 + (1 - 2\operatorname{cap}(E))^{1/2}}{1 - (1 - 2\operatorname{cap}(E))^{1/2}} \right).$$

Next we present an analogue of results above for complex polynomials. Let $|A|$ be the linear measure (length) of a set $A$ in the complex plane $\mathbb{C}$. By $\mathbb{P}_n$ we denote the set of all complex polynomials of degree at most $n \in \mathbb{N}$. Let

$$\Pi(p) := \{z \in \mathbb{C} : |p(z)| > 1\}, \qquad p \in \mathbb{P}_n.$$

From the numerous generalizations of the Remez inequality, we cite one result which is a direct consequence of the trigonometric version of the Remez inequality (and is equivalent to this trigonometric version, up to constants).

Assume that $p \in \mathbb{P}_n$, $\mathbb{T} := \{z : |z| = 1\}$ and

$$|\mathbb{T} \cap \Pi(p)| \leq s, \quad 0 < s \leq \frac{\pi}{2}. \tag{6.8}$$

Then, $q(t) := |p(e^{it})|^2$ is a trigonometric polynomial of degree at most $n$ and, by the Remez-type inequality on the size of trigonometric polynomials (cf. [21, Theorem 2], [14, p. 230]), we have

$$||p||_{\mathbb{T}} \leq e^{2sn}, \quad 0 < s \leq \frac{\pi}{2}. \tag{6.9}$$

Here, $||\cdot||_A$ means the uniform norm along $A \subset \mathbb{C}$.

Our next aim is to discuss an analogue of (6.8)-(6.9) in which we use logarithmic capacity instead of the length. As before our main result deals not only with polynomials, but also with exponentials of potentials.

**Theorem 21.** *([3]) Let $0 < \delta < 1$. Then, the condition*

$$\operatorname{cap}(E_{\nu,c}) \geq \delta$$

*implies that*

$$||Q_{\nu,c}||_{\mathbb{T}} \leq \left( \frac{1+\sqrt{1-\delta^2}}{\delta} \right)^{\nu(\mathbb{C})}.$$

**Remark 5** In order to examine the sharpness of Theorem 21 we consider the following example.

Let $0 < \alpha < \pi/2$, and let $L = L_\alpha := \{e^{i\theta} : 2\alpha \leq \theta \leq 2\pi - 2\alpha\}$. Since the function

$$z = \Psi(w) = -w \frac{w-a}{1-aw},$$

where $a = 1/\cos\alpha$, maps $\Delta$ onto $\Omega := \overline{\mathbb{C}} \setminus L$ (cf. [26]) and since the Green function of $\Omega$ with pole at $\infty$ can be defined via the inverse function $\Phi := \Psi^{-1}$ by the formula

$$g_\Omega(z) = \log|\Phi(z)|, \quad z \in \Omega,$$

we have

$$\operatorname{cap}(L) = \lim_{w\to\infty} \frac{\Psi(w)}{w} = \frac{1}{a} = \cos\alpha, \tag{6.10}$$

as well as

$$\begin{aligned} \max_{z\in\mathbb{T}\setminus L} g_\Omega(z) &= g_\Omega(1) = \log|\Phi(1)| \\ &= \log(a+\sqrt{a^2-1}) = \log\frac{1+\sqrt{1-\operatorname{cap}(L)^2}}{\operatorname{cap}(L)}. \end{aligned} \tag{6.11}$$

Let $c = c_\alpha := -\log\operatorname{cap}(L)$ and let $\nu = \nu_\alpha$ be the equilibrium measure for $L$; that is, $\nu(\mathbb{C}) = 1$. Since for $z \in \mathbb{C}$,

$$U^\nu(z) = -g_{\overline{\mathbb{C}}\setminus L}(z) - \log\operatorname{cap}(L),$$

and therefore

$$Q_{\nu,c}(z) = \exp\left(g_{\overline{\mathbb{C}}\setminus L}(z)\right),$$

we have $E_{\nu,c} = L$ as well as

$$||Q_{\nu,c}||_{\mathbb{T}} = \frac{1+\sqrt{1-\operatorname{cap}(L)^2}}{\operatorname{cap}(L)}.$$

This shows the exactness of Theorem 21.

**Remark 6** Let $p(z) = c\prod_{j=1}^{n}(z-z_j)$, $0 \neq c \in \mathbb{C}$, be a complex polynomial of degree $n$, and let

$$\nu_n := \sum_{j=1}^{n} \delta_{z_j}.$$

For $z \in \mathbb{C}$, we have

$$Q_{\nu_n, \log|c|} = \exp\ (\log|c| + \log\prod_{j=1}^{n}|z-z_j|) = |p(z)|.$$

Therefore, applying the above theorem, we obtain the following, for $0 < \delta < 1$: For $p \in \mathbb{P}_n$ the condition

$$\mathrm{cap}(\mathbb{T}\setminus \Pi(p)) \geq \delta \tag{6.12}$$

implies

$$||p||_{\mathbb{T}} \leq \left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^n. \tag{6.13}$$

**Remark 7** Since for any $E \subset \mathbb{T}$ we have $\mathrm{cap}(E) \geq \sin\frac{|E|}{4}$ (see [37]), (6.12)-(6.13) imply the following refinement of (6.8)-(6.9): For $p \in \mathbb{P}_n$ the condition

$$|\mathbb{T}\cap \Pi(p)| \leq s, \quad 0 < s < 2\pi, \tag{6.14}$$

implies

$$||p||_{\mathbb{T}} \leq \left(\frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}\right)^n. \tag{6.15}$$

This result is also sharp in the following sense. Let $0 < s < 2\pi$, $\alpha = s/4$, and let $L = L_\alpha$ be defined as in Remark 5. By (6.10) and (6.11),

$$g_\Omega(1) = \log\frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}.$$

We denote by $f_n(z)$ the $n$-th Fekete polynomial for a compact set $L$ (see [41]). Hence, condition (6.14) holds for the polynomial $p(z) = p_n(z) := f_n(z)/||f_n||_L$. At the same time, since

$$\lim_{n\to\infty}\left(\frac{|f_n(z)|}{||f_n||_L}\right)^{1/n} = \exp(g_\Omega(z)), \quad z \in \Omega\setminus\{\infty\}$$

(see [41, p. 151]), we have

$$\lim_{n\to\infty}|p(1)|^{1/n} = \frac{1+\sin\frac{s}{4}}{\cos\frac{s}{4}}.$$

Theorem 21 is a straightforward consequence of its following particular case.

**Lemma 2.** *([3]) Let $E \subset \mathbb{T}$ be a compact set with $0 < \operatorname{cap}(E) < 1$. Then,*

$$\sup_{z\in\mathbb{T}\setminus E} g_{\overline{\mathbb{C}}\setminus E}(z) \le \log\left(\frac{1+\sqrt{1-\operatorname{cap}(E)^2}}{\operatorname{cap}(E)}\right).$$

In our approach, we exploit the following simple connection between estimates which express the possible growth of a polynomial with a known norm on a given compact set $E \subset \mathbb{C}$ and the behavior of the Green's function for $\overline{\mathbb{C}} \setminus E$.

Let the logarithmic capacity of $E$ be positive and let $\Omega := \overline{\mathbb{C}} \setminus E$ be connected. For $z \in \Omega$ and $u > 0$, the following two conditions are equivalent:

(i) $g_\Omega(z) \le u$;

(ii) for any $p \in \mathbb{P}_n$ and $n \in \mathbb{N}$,

$$|p(z)| \le e^{un}||p||_E.$$

Indeed, (i) $\Rightarrow$ (ii) follows from the Bernstein–Walsh lemma and (ii) $\Rightarrow$ (i) is a simple consequence of a result by Myrberg and Leja (see [35, p. 333]).

We study the properties of the Green function by methods of geometric function theory (involving symmetrization) which allows, according to the implication (i) $\Rightarrow$ (ii), to obtain results similar to (6.12) - (6.13).

Note that the sharpness of results for the Green function, by virtue of the equivalence of (i) and (ii), implies the (asymptotic) sharpness of the corresponding Remez-type inequalities for polynomials (6.12) - (6.13).

## References

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