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Pseudo-nearrings and quasi-modules over them

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Abstract. In this paper we start to investigate a new notion of pseudo-nearrings and a generalization of linear spaces to quasi-modules over pseudo-nearrings. Pseudo-nearrings can be treated as ringoids in the sense of J. Hion (see [6]). The idea of pseudo-nearings is based on the notion of a *-associative quasigroup, i.e. on an involutive groupoid $(A;+,^*)$ in which the following identities hold:

$$(x^*)^* = x,\ (x+y)^* = y^* + x^*,\ (x+y)^* + z = x + (y+z)^*.$$

We assume also commutativity and quasigroup properties of $(A;+)$.

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1. Introduction

An algebra $(A;+,^*)$ is said to be an *involutive groupoid* if the following identities hold:

$$(x^*)^* = x, \quad (x+y)^* = y^* + x^*.$$

We call an involutive groupoid *-*associative* if it satisfies the equation:

$$(x+y)^* + z = x + (y+z)^*.$$

A *-associative groupoid $(A;+,^*)$ is a *-*associative quasigroup* if $(A;+)$ is a quasigroup.

The concepts of *- associative groupoid and quasigroup were introduced in [2]. For the standard terminology of semigroups, quasigroups and near-rings, see [1], [7], [9] and [10].

Examples 1 and 2. Define the following operations in the set Z_5:

$$x^* = 4x(mod 5), x \oplus y = 4x + 4y + 2x^2y^2(x+y)(mod 5);$$

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and in the set Z_7:

$$x^* = 6x(mod7), x \oplus y = 6x + 6y + 2x^2y^2(x^3 + y^3) + 4x^3y^3(x + y)(mod7).$$

Then $(Z_5; \oplus, ^*)$ and $(Z_7; \oplus, ^*)$ are *-associative quasigroups.

Example 3. The algebra $(Z; \oplus, ^*)$, where Z is the set of integers, $x \oplus y = -(x+y)+3a$ (for some fixed $a \in Z$), and $x^* = -x+2a$, is a *-associative quasigroup.

For more examples, we refer the reader to [2] and [5].

In our investigation, we use the following proposition (see [3]):

Proposition 1. *Let $(A; +, ^*)$ be a commutative *-associative groupoid. Then*

(a) $(a + b) + c = (a + c^*) + b^*$;

(b) $(a + b) + (c + d) = (a + c) + (b + d)$, *for all* $a, b, c \in A$.

Therefore the considered groupoid $(A; +)$ is *medial.*

In the following we need to have a description of term operations for the considered algebras.

For a fixed algebra $\mathcal{A} = (A; \mathbb{F})$ and $n = 0, 1, 2, \ldots$, we denote by $\mathbb{T}^{(n)}(\mathcal{A})$ (or $\mathbb{T}^{(n)}$ for short) the class of all n-ary term operations of $\mathcal{A}$, i.e. the smallest class of operations satisfying the conditions:

(i) $e_i^n \in \mathbb{T}^{(n)}$, $\quad (e_i^n(x_1, x_2, \ldots, x_n) = x_i$ for $i = 1, 2, \ldots, n)$

(ii) if $g_1, g_2, \ldots, g_k \in \mathbb{T}^{(n)}$, $f \in \mathbb{F}^{(k)}$, then

$$f(g_1, g_2, \ldots, g_k)(x_1, x_2, \ldots, x_n) = \\ = f(g_1(x_1, x_2, \ldots, x_n), \ldots, g_k(x_1, x_2, \ldots, x_n))$$

belongs to $\mathbb{T}^{(n)}$.

For completeness, we recall the description of term operations in commutative *-associative groupoids (see [3]).

Denote $g_{(k,l)}(x) = (\ldots(x^{i_1} + x^{i_2}) + \ldots) + x^{i_l}) + \ldots) + x^{i_k}$, where $k \in N$, $l \in N \cup \{0\}$, l is the first place in which * appears, $x \in A$, $x^0 = x^*$, $x^1 = x$, i.e., if $l = 1$, then $i_1 = 0$, $i_{2r} = 0, i_{2r+1} = 1$ for $r \in N$, and if $l > 1$, then $i_r = 1$ for $r < l$; $i_{l+2s} = 0, i_{l+2s+1} = 1$ for $r, s \in N \cup \{0\}$.

Then we have (see [3]):

Theorem 1. *Every unary term operation in a commutative *-associative groupoid $(A; +, ^*)$ is of the form $g_{(k,l)}(x)$.*

Define $h_{(p,r)}(y,x) = (\ldots(y+x^{i_1})+x^{i_2})+\ldots)+x^{i_r})+\ldots)+x^{i_p}$, where $p, r \in N \cup \{0\}$, $h_{(0,0)}(y,x) = y$, $i_s = 0$ for $s < r$; $i_{r+2s} = 0, i_{r+2s+1} = 1$ for $s \in N \cup \{0\}$.

For simplicity of notation, we write $g_{(k,l)}(x) \oplus h_{(p,r)}(x)$ instead of $h_{(p,r)}(g_{(k,l)}(x), x)$. We obtain (see [3]):

Proposition 2. *Every n-ary term operation in a commutative *-associative groupoid $(A, +,^*)$ is of the following form:*

$f(x_1, x_2, \ldots, x_n) = (\ldots(g_{(k_1,l_1)}(x_1) \oplus h_{(k_2,l_2)}(x_2)) \oplus \ldots) \oplus h_{(k_n,l_n)}(x_n)$.

2. *-Associative quasigroups

We have the following characterization of *-associative quasigroups (see [2]):

Theorem 2. *Let $(A; +,^*)$ be a *-associative groupoid. Then $(A; +)$ is a quasigroup if and only if the following two conditions hold:*

(i) $(\exists \varepsilon \in A)\ (\forall a \in A) \quad \varepsilon + a = a^*$,

(ii) $(\forall a \in A)\ (\exists b \in A) \quad b + a = \varepsilon$.

Let $\mathcal{A} = (A; +,^*, \varepsilon)$ be a *-associative quasigroup. Define $-a$ as an element such that $a + (-a) = \varepsilon$. To shorten notation, we write $a - b$ instead of $a + (-b)$. The unary operation $a \mapsto -a$ can be added to the set of fundamental operations of $\mathcal{A}$. Now, we prove several simple properties of *-associative quasigroups.

Proposition 3. *Let $(A; +, -,^*, \varepsilon)$ be a *-associative quasigroup. Then*

(a) $(-a)^* = -(a^*)$;

(b) $-(a + b) = (-b) + (-a)$;

(c) $a = b \Leftrightarrow a - b = \varepsilon$; *and*

(d) $[a = b + c] \Leftrightarrow a - b^* = c$, *for all* $a, b, c \in A$.

Proof. The first equality is obvious. For the second we have:

$$\begin{aligned}(a + b) + ((-b) + (-a)) &= (a + b) + ((-a^*) + (-b^*))^* = \\ &= ((a + b) + (-a^*))^* + (-b^*) = ((b^* + a^*)^* + (-a^*))^* + (-b^*) = \\ &= (b^* + (a^* + (-a^*))^*)^* + (-b^*) = (b^* + \varepsilon)^* + (-b^*) = b^* + (-b^*) = \varepsilon.\end{aligned}$$

The proofs of the last two equivalences are straightforward. □

If there exists an idempotent e (i.e., $e{+}e = e, e^* = e$) in a *-associative groupoid $(A;+,^*)$, then we can define a set Q_e as follows:

$$Q_e = \{a \in A : e + a = a + e = a^*,\ a + b = b + a = e \text{ for some } b \in A\}.$$

Proposition 4. *Let $\mathcal{A} = (A;+,^*)$ be a *-associative groupoid. Then Q_e is a *-associative quasigroup and*

$$Q_e = \{a \in A\ :\ \ a^* \in (e + A) \cap (A + e),\ e \in (a + A) \cap (A + a)\}.$$

Proof. We first prove that Q_e is a subalgebra of $\mathcal{A}$. Let $a \in Q_e$. Then $a + e = e + a = a^*$ and $a + b = b + a = e$ for some $b \in A$. This yields $e + a^* =(a + e)^* = (a^*)^* = a = a^* + e,\ \ a^* + b^* = (b + a)^* = e = b^* + a^*$. This clearly forces $a^* \in Q_e$.

Now, suppose that $a, b \in Q_e$. Therefore $a + c = c + a = e$ and $b + d = d + b = e$ for some $c, d \in A$. Then

$$e + (a + b)^* = (e + a)^* + b = a^{**} + b = a + b,$$

$$\begin{aligned}(a + b)^* + e = a + (b + e)^* = a + b, (a + b)^* + (d + c)^* =\\ = a + (b + (d + c)^*)^* = a + ((b + d)^* + c)^* = a + (e + c)^* =\\ = (a + e)^* + c = a + c = e = (d + c)^* + (a + b)^*.\end{aligned}$$

This implies $(a + b)^* \in Q_e$, and consequently $a + b \in Q_e$.

To prove that $Q_e \supseteq \{a \in A :\ a^* \in (e + A) \cap (A + e),\ e \in (a + A) \cap (A + a)\}$, let $a^* \in (e + A) \cap (A + e)$ and $e \in (a + A) \cap (A + e)$. Hence $a^* = e + p$, $a^* = q + e$, $e = a + r$ and $e = t + a$ for some $p, q, r, t \in A$. So $e + a = e + (e + p)^* = (e + e)^* + p = e + p = a^*$, and analogously $a + e = a^*$.

Now, put $b = (t + e)^*$. Then $t + e = t + e^* = t + (a + r)^* = (t{+}a)^*{+}r = e{+}r = (t{+}a){+}r$. Thus $(t{+}a){+}r = b^* = t{+}(a{+}r)$ and so $a{+}b = a{+}((t{+}a){+}r)^* = a{+}(e{+}r)^* = (a{+}e)^*{+}r = a^{**}{+}r = a{+}r = e$. In the same manner, we can see that $b{+}a = e$. The result is $a \in Q_e$. □

Here and subsequently, we denote $i = 0, 1,\ j = i + 1 (mod\ 2)$.

Lemma 1. *The following properties hold in every commutative *-associative quasigroup $(A;+,-,^*,\varepsilon)$:*

(a) $-(a - a^*) = (a - a^*)^*$,

(b) $a^i + (a - a^*)^i = a^i$,

(c) $(a + a^*) + (a - a^*)^i = a^j + a^j$,

Proof. It is immediate. □

Let $g_{(k,l)}(x)$ be as in Section 1, and let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote the sets of all terms of the form $g_{(k,l)}(x)$ fulfilling the following conditions:

1) $l = 0$ for $k = 2$, $l = 1$ for k odd, $l = 3$ for $k > 2$, k even;

2) $l = 0$ for $k = 1$, $l = 1$ for k even, $l = 3$ for $k > 1, k$ odd, respectively.

Denote

$$f_{(k,l,m,n)}(x) = g_{(k,l)}(x) + g_{(m,n)}(x - x^*),$$

where $k \in N$, $l, m,n \in N \cup \{0\}$, and $g_{(0,0)}(x) = \varepsilon$.

If $m \neq 0$, then $[g_{(k,l)}(x) \in \mathbb{T}_1$ and $g_{(m,n)}(x) \in \mathbb{T}_2]$ or $[g_{(k,l)}(x) \in \mathbb{T}_2$ and $g_{(m,n)}(x) \in \mathbb{T}_1]$.

Theorem 3. *Every unary term operation in a commutative* **-associative quasigroup* $(A; +, -,^*, \varepsilon)$ *is of the form* $\pm f_{(k,l,m,n)}(x)$.

Proof. Obviously, every *-associative quasigroup is also a *-associative groupoid. So, by Theorem 1, the term operations of the form $g_{(k,l)}(x)$ and also $-g_{(k,l)}(x)$ belong to the set of term operations of a commutative *-associative quasigroup.

From Propositions 1(b), 3(a) and 3(b) we deduce that the term operations are also of the form

$$g_{(k,l)}(x) \pm g_{(m,n)}(x - x^*) \text{ or } -g_{(k,l)}(x) \pm g_{(m,n)}(x - x^*).$$

By Lemma 1(a), these forms are equivalent to

$$\pm[g_{(k,l)}(x) + g_{(m,n)}(x - x^*)] = \pm f_{(k,l,m,n)}(x).$$

We consider only the term operations of the form $f_{(k,l,m,n)}(x)$, because for $-f_{(k,l,m,n)}(x)$ the verification is similar.

We first observe that, for $m \neq 0$, it is enough to consider the terms $g_{(k,l)}(x)$ which belong to $\mathbb{T}_1$ or $\mathbb{T}_2$, because the other forms of term operations can be rewritten in a suitable form, i.e. $f_{(k',l',m',n')}(x)$.

For $k = 2$, by Lemma 1(c) we have

$$\begin{aligned} f_{(2,2,m,n)}(x) &= (x + x^*) + g_{(m,n)}(x - x^*) = \\ &= (x + x^*) + [g_{(m-1,n)}(x - x^*) + (x - x^*)^i] = \\ &= [(x + x^*) + (x - x^*)^i] + g^*_{(m-1,n)}(x - x^*) = \\ &= (x^j + x^j) + g^*_{(m-1,n)}(x - x^*), \end{aligned}$$

where $x^j + x^j = g_{(2,i)}(x) \in \mathbb{T}_1$ or $\mathbb{T}_2$.

For $k > 2$ and k even, by Lemma 1(b), we obtain

$$\begin{aligned}
f_{(k,2,m,n)}(x) = g_{(k,2)}(x) + g_{(m,n)}(x - x^*) = \\
[(x + x^*) \oplus g_{(k-2,2)}(x)] + [g_{(m-1,n)}(x - x^*) + (x - x^*)^i] = \\
[(x^j \oplus g_{(k-2,2)}(x)) + x^i] + [g_{(m-1,n)}(x - x^*) + (x - x^*)^i] = \\
[x^i + (x - x^*)^i] + [(x^j \oplus g_{(k-2,2)}(x)) + g_{(m-1,n)}(x - x^*)] = \\
x^i + [(x^j \oplus g_{(k-2,2)}(x)) + g_{(m-1,n)}(x - x^*)] = \\
[x^j + (x^j \oplus g_{(k-2,2)}(x)] + g^*_{(m-1,n)}(x - x^*) = \\
[(x^j + x^j) \oplus g_{(k-2,2)}(x)] + g^*_{(m-1,n)}(x - x^*).
\end{aligned}$$

After a finite number of similar steps, we get the term operation of the form $f_{(k',l',m',n')}(x)$, where $m' = 0$ or $g_{(k',l')}(x)$ belongs to $\mathbb{T}_1$ or $\mathbb{T}_2$. The same conclusion can be drawn for k odd. Therefore we can assume $l \neq 2$.

Applying the equality $(x+x)+x = (x+x^*)+x^*$, by the same method as before, we can see that $l \leq 3$ for $k \geq 3$. Consequently, $g_{(k,l)}(x) \in \mathbb{T}_1$ or $g_{(k,l)}(x) \in \mathbb{T}_2$.

Since

$$g_{(2,2)}(x - x^*) = (x - x^*) + (x - x^*)^* = \varepsilon$$

and

$$g_{(3,0)}(x - x^*) = ((x - x^*) + (x - x^*)) + (x - x^*) = (x - x^*),$$

we conclude that the term operations of the form $g_{(m,n)}(x - x^*)$ belong to $\mathbb{T}_1$ or $\mathbb{T}_2$.

Now, let $g_{(k,l)}(x), g_{(m,n)}(x) \in \mathbb{T}_1$. Then for $k = 1$, by Lemma 1(b), we get

$$\begin{aligned}
f_{(1,1,m,n)}(x) = \\
g_{(1,1)}(x) + g_{(m,n)}(x - x^*) = x^* + [g_{(m-1,n)}(x - x^*) + (x - x^*)] = \\
[x + (x - x^*)] + g^*_{(m-1,n)}(x - x^*) = x + g^*_{(m-1,n)}(x - x^*) = \\
g_{(1,0)}(x) + g^*_{(m-1,n)}(x - x^*),
\end{aligned}$$

where $g_{(1,0)}(x) \in \mathbb{T}_2$ and $g^*_{(m-1,n)}(x - x^*) \in \mathbb{T}_1$.

For $k > 1, m = 1$, we see that

$$\begin{aligned}
f_{(k,l,1,1)}(x) = g_{(k,l)}(x) + g_{(1,1)}(x - x^*) = \\
[g_{(k-1,l)}(x) + x] + (x - x^*)^* = [x + (x - x^*)] + g^*_{(k-1,l)}(x) = \\
g^*_{(k-1,l)}(x) + x.
\end{aligned}$$

So, we can rewrite this term operation in the form $f_{(k',l',m',n')}(x)$, where $m' = 0$.

For $k = 2$ and $m = 2$, by Proposition 1(b) and Lemma 1(b), we have $f_{(2,0,2,0)}(x) = g_{(2,0)}(x)$.

For $k = 2$ and $m > 2$, we obtain

$$\begin{aligned}
f_{(2,0,m,n)}(x) = [x + x] + [g_{(m-1,n)}(x - x^*) + (x - x^*)] = \\
[x + (x - x^*)] + [x + g_{(m-1,n)}(x - x^*)] = x + [x + g_{(m-1,n)}(x - x^*)] = \\
[x + x^*] + g^*_{(m-1,n)}(x - x^*) = [x + x^*] + [g^*_{(m-2,n)}(x - x^*) + (x - x^*)] = \\
[(x + x^*)^* + (x - x^*)] + g_{(m-2,n)}(x - x^*) = [(x + x^*) + (x - x^*)] + \\
g_{(m-2,n)}(x - x^*) = [x^* + x^*] + g_{(m-2,n)}(x - x^*),
\end{aligned}$$

where $g_{(2,1)}(x) \in \mathbb{T}_2$ and $g_{(m-2,n)}(x - x^*) \in \mathbb{T}_1$.

For $k > 2, m \geq 2$, we get

$$\begin{aligned}
f_{(k,l,m,n)}(x) = \\
[(g_{(k-2,l)}(x) + x^*) + x] + [(g_{(m-2,n)}(x - x^*) + (x - x^*)^*) + (x - x^*)] = \\
[g^*_{(k-2,l)}(x) + (x^* + x^*)] + [g^*_{(m-2,n)}(x - x^*) + ((x - x^*)^* + (x - x^*)^*] = \\
[g^*_{(k-2,l)}(x) + g^*_{(m-2,n)}(x - x^*)] + [(x^* + x^*) + ((x - x^*)^* + (x - x^*)^*)] = \\
[g^*_{(k-2,l)}(x) + g^*_{(m-2,n)}(x - x^*)] + (x^* + x^*) = \\
[(x + x) + g^*_{(k-2,l)}(x)] + g_{(m-2,n)}(x - x^*),
\end{aligned}$$

so $m' = 0$ or after a finite number of similar steps, we get the term operation of the required form.

The same conclusion can be drawn for the case $g_{(k,l)}(x), g_{(m,n)}(x) \in \mathbb{T}_2$.

□

3. Pseudo-nearrings

A *pseudo-nearrning* is an algebra $(A; +, \cdot, ^*, \eta)$ of the type $(2, 2, 1, 0)$ fulfilling the following conditions:

(i) $(A; +, ^*, \eta)$ is a commutative *-associative quasigroup,

(ii) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,

(iii) $(\alpha\beta)^* = \alpha^*\beta$,
and

(iv) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$, for all $\alpha, \beta, \gamma \in A$.

Example 4. Let $(A; +,^*, \eta)$ be a *-associative quasigroup and $\mathcal{T}_A$ be the set of all maps from A to itself. Then the algebra $(\mathcal{T}_A; \#, \circ,^{\otimes}, f_\eta)$, where $(f\#g)(x) = f(x) + g(x)$, $(f \circ g)(x) = f(g(x))$, $f^{\otimes}(x) = (f(x))^*$ and $f_\eta(x) = \eta$, is a pseudo-nearring.

Indeed, it is easy to show that $(\mathcal{T}_A; \#,^{\otimes})$ is a commutative *-associative groupoid. We prove that it is a quasigroup. Let $h \in \mathcal{T}_A$. Then $(f_\eta\#h)(x) = f_\eta(x) + h(x) = \eta + h(x) = (h(x))^* = h^{\otimes}(x)$, which gives $f_\eta\#h = h^{\otimes}$.

By Lemma 2 of [2], if $a \in A$, then there exists a unique element b, such that $a + b = \eta = b + a$. Define a map $g_\eta : A \mapsto A$ as follows $g_\eta(x) = y \Leftrightarrow x + y = \eta$.

Let $h \in \mathcal{T}_A$. Then $((g_\eta \circ h)\#h)(x) = g_\eta(h(x)) + h(x) = \eta = f_\eta(x)$, and, in consequence, $(g_\eta \circ h)\#h = f_\eta$.

Example 5. Let $(Z; \oplus,^*)$ be the *-associative quasigroup defined in Example 3 and a multiplication be given by $x \circ y = a$. Then $(Z; \oplus, \circ,^*, a)$ is a pseudo-nearring.

Example 6. Let $(A; +,^*, \eta)$ be a *-associative quasigroup. Define the operation $x \circ y = x$. Then $(A; +, \circ,^*, \eta)$ is also a pseudo-nearring.

Proposition 5. *Let $(A; +, \cdot,^*, \eta)$ be a pseudo-nearring. Then*

$$\eta\alpha = \eta \text{ and } -(\alpha\beta) = (-\alpha)\beta.$$

Proof. Suppose that $\alpha \in A$. By Theorem 2 and (iii), (iv) from the pseudo-nearring definition, we have:

$$\eta = \eta\alpha + (-\eta\alpha) = (\eta^* + \eta^*)\alpha + (-\eta\alpha) = (\eta\alpha + \eta\alpha)^* + (-(\eta\alpha)) = \\ = \eta\alpha + (\eta\alpha + (-(\eta\alpha))^* = \eta\alpha.$$

The proof of the second property is immediate. □

We use similar notations as in [10]. Define two subsets of a pseudo-nearring $(A; +, \cdot,^*, \eta)$. A set $A_\eta = \{a \in A : a\eta = \eta\}$ is called the *η-symmetric part* of $\mathcal{A}$ and $A_c = \{a \in A : a\eta = a\}$ is called the *constant part* of $\mathcal{A}$. It is evident that A_η, A_c are subalgebras of $\mathcal{A}$.

Proposition 6. *Let $(A; +, \cdot,^*, \eta)$ be a pseudo-nearring. Then*

$$(\forall a \in A)\ (\exists a_\eta \in A_\eta)\ (\exists a_c \in A_c)\ \ a = a_\eta + a_c.$$

Proof. Similarly as for nearrings the element $a = (a^* + (-a^*\eta))^* + a^*\eta$ will do the decomposition job. □

A non-empty subset I of a pseudo-nearring $\mathcal{A}$ is said to be a *left ideal of* $\mathcal{A}$ if:

$$(\forall \alpha, \beta \in I)\ (\forall \gamma \in A) \quad [\gamma\alpha \in I, \alpha^* \in I, \alpha - \beta \in I].$$

Remark. The subset A_c is a left ideal of $\mathcal{A}$.

4. Quasi-modules over pseudo-nearrings

Let $(V; +,^*, \varepsilon)$ be a commutative *-associative quasigroup and $\mathcal{A} = (A; \oplus, \circ,^\star, \eta)$ be a pseudo-nearring. Define a map $f_\alpha : V \to V;\ u \longmapsto \alpha u$ for all $\alpha \in A$. If, for all $\alpha, \beta \in A$; and for all $u, v \in V$, we have:

(i) $(\alpha \oplus \beta)u = \alpha u + \beta u$,

(ii) $\alpha(u + v) = \alpha u + \alpha v$,

(iii) $(\alpha \circ \beta)u = \alpha(\beta u)$,

(iv) $\alpha\varepsilon = \varepsilon = \eta u$,
and

(v) $\alpha u = \beta u \Rightarrow \alpha = \beta$,

then V with operations $+,^*, \varepsilon, (f_\alpha)_{\alpha \in A}$ is called a *quasi-module over* $\mathcal{A}$.

Proposition 7. *Let* $(V, +,^*, \varepsilon, (f_\alpha)_{\alpha \in A})$ *be a quasi-module over a pseudo-nearring* $(A; \oplus, \circ,^\star, \eta)$ *and* $\alpha \in A, u \in V$*. Then we have:*

(a) $\alpha^\star u = (\alpha u)^* = \alpha u^*$;

(b) $-(\alpha u) = (-\alpha)u = \alpha(-u)$;

(c) $\alpha u = \varepsilon \Rightarrow [\alpha = \eta$ *or* $u = \varepsilon]$.

Proof. Let $\alpha \in A, u \in V$. Then

$$\alpha^\star u = (\alpha \oplus \eta)u = \alpha u + \eta u = \alpha u + \varepsilon = (\alpha u)^* = \alpha u + \alpha\varepsilon = \alpha(u + \varepsilon) = \alpha u^*.$$

The rest of the proof is standard. □

Denote

$$F_{(k,l,m,n,\alpha)}(x) = f_{(k,l,m,n)}(x) + \alpha x,$$

where $f_{(k,l,m,n)}(x)$ is defined as in Section 2, $\alpha \in A, x \in V$.

Theorem 4. *Every unary term operation in a quasi-module* $(V, +,^*, \varepsilon, (f_\alpha)_{\alpha\in A})$ *over a pseudo-nearring* $(A, \oplus, \circ,^\star, \eta)$ *is of the form* $\pm F_{(k,l,m,n,\alpha)}(x)$.

Proof. The proof is by induction with respect of the complexity of operations. We first observe that the projection has the required form:

$$e_1^1(x) = x = (x+\varepsilon) + \eta x = F_{(1,0,0,0,\eta)}(x).$$

The set of all operations of the form $\pm F_{(k,l,m,n,\alpha)}(x)$ is closed under the quasi-module operations. Indeed, by Propositions 3(b) and 7(b), we conclude that

$$-F_{(k,l,m,n,\alpha)}(x) = -f_{(k,l,m,n)}(x) + (-\alpha)x.$$

Taking into account Proposition 1(b), we get

$$F_{(k_1,l_1,m_1,n_1,\alpha_1)}(x) + F_{(k_2,l_2,m_2,n_2,\alpha_2)}(x) = [f_{(k_1,l_1,m_1,n_1)}(x) + f_{(k_2,l_2,m_2,n_2)}(x)] + (\alpha_1+\alpha_2)x.$$

From Proposition 7(a), it follows that

$$F^*_{(k,l,m,n,\alpha)}(x) = f^*_{(k,l,m,n)}(x) + \alpha^\star x.$$

Now, we verify that

$$\beta F_{(k,l,m,n,\alpha)}(x) = \beta[f_{(k,l,m,n)}(x) + \alpha x] = \beta(g_{(k,l)}(x) + g_{(m,n)}(x - x^*)) + (\beta\alpha)x = (\beta g_{(k,l)}(x) + \beta g_{(m,n)}(x-x^*)) + (\beta\alpha)x.$$

By parts (i) and (ii) of the quasi-module definition, and Proposition 7(a), we have

$$\beta g_{(k,l)}(x) = \beta[\ldots(x^{i_1} + x^{i_2}) + \ldots) + x^{i_l}) + \ldots) + x^{i_k}] = (\ldots(\beta x^{i_1} + \beta x^{i_2}) + \ldots) + \beta x^{i_l}) + \ldots) + \beta x^{i_k} = (\ldots(\beta^{i_1}x + \beta^{i_2}x) + \ldots) + \beta^{i_l}x) + \ldots) + \beta^{i_k}x = [(\ldots(\beta^{i_1} + \beta^{i_2}) + \ldots) + \beta^{i_l}) + \ldots) + \beta^{i_k}]x = g_{(k,l)}(\beta)x.$$

And also $\beta g_{(m,n)}(x - x^*) = g_{(m,n)}(\beta)(x-x^*) = [g_{(m,n)}(\beta) - g^*_{(m,n)}(\beta)]x$. Here, we use similar notations for term operations $g_{(k,l)}(x)$ in a quasi-module as for term operations $g_{(k,l)}(\beta)$ in a pseudo-nearring.

Finally, we deduce that

$$\beta F_{(k,l,m,n,\alpha)}(x) = [g_{(k,l)}(\beta)x + (g_{(m,n)}(\beta) - g^*_{(m,n)}(\beta))x] + (\beta\alpha)x = [(g_{(k,l)}(\beta) + (g_{(m,n)}(\beta) - g^*_{(m,n)}(\beta))) + (\beta\alpha)]x = F_{(0,0,0,0,\gamma)}(x),$$

where $\gamma = [(g_{(k,l)}(\beta) + (g_{(m,n)}(\beta) - g^*_{(m,n)}(\beta))) + (\beta\alpha)]^*$. □

In the next paper we will use the obtained results, especially the description of term operations in *-associative quasigroups and quasimodules over pseudo-nearrings, for the description of independent sets (in the sense of Marczewski; see [8]) in the above algebras.

References

[1] I. Chajda, K. Głazek, *A Basic Course on General Algebra*, Technical Univ. Press, Zielona Góra 2000.

[2] A. Chwastyk, K. Głazek, *Remarks on *-associative groupoids*, Contributions to General Algebra **13** (2001), 83-89.

[3] A. Chwastyk, K. Głazek, *Term operations in commutative *-associative groupoids*, Contributions to General Algebra **14** (2003), in print.

[4] K. Głazek, *On some non-associative rings* (in Polish), Acta Univ. Wratislav. **17** (1961), 15-19.

[5] K. Głazek, **-associative and γ-algebras* (in Polish), Acta Univ.Wratislav. **58** (1967), 5-19.

[6] J. Hion, *Ω-ringoids, Ω-rings and their representations* (in Russian), *Trudy Moskov. Mat. Obshch.* **14** (1965), 3-47.

[7] A.G. Kurosh, *General Algebra Lectures of the 1969-1970 Academic Year* (in Russian), Izdat. "Nauka", Moscow 1974.

[8] E. Marczewski, *Independence and homomorphism in abstract algebras*, Fund. Math. **50** (1961), 45-61.

[9] H.O. Pflugfelder, *Quasigroups and Loops: Introduction*, Heldermann-Verlag, Berlin 1990.

[10] G. Pilz, *Near-Rings*, North-Holland Publ. Comp., Amsterdam 1983.

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