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GORIACHEV'S SOLUTION IN THE DYNAMICS OF A RIGID BODY ABOUT A FIXED POINT

A natural and analytic parametrization of the expressions for the Euler–Poisson variables in the case found by Goriachev in 1899 is constructed. The solution given by Stepanova in 1974 is not adequate for description of motion at all times.

Keywords: *Goriachev's solution, elliptic functions, rigid body.*

Introduction and history. The dynamics of the heavy rigid body about a fixed point is described by the system of Euler–Poisson equations. That is composed of six nonlinear equations, usually written in the form

$$\begin{gathered} A\dot{p} + (C - B)qr = Mg(z_0\gamma_2 - y_0\gamma_3), \\ B\dot{q} + (A - C)\,pr = Mg(x_0\gamma_3 - z_0\gamma_1), \\ C\dot{r} + (B - A)pq = Mg(y_0\gamma_1 - x_0\gamma_2), \\ \dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \quad \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \quad \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0 \end{gathered} \tag{1}$$

in which dots represent differentiation with respect to time t, $(p, q, r), (\gamma_1, \gamma_2, \gamma_3)$ and (x_0, y_0, z_0) are, respectively, the angular velocity of the body, the unit vector directed vertically upward and the position vector of the centre of mass of the body, all referred to the system of principal axes of inertia of the body at the fixed point, M is the mass of the body and g is the acceleration of gravity. This system admits the integrals of motion

$$\begin{aligned} I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h, \\ I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f, \\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{2}$$

To be integrable, the above system has to admit an independent fourth integral. For a whole century, only two cases with this property were known: the cases bearing the names of Euler [1] and Lagrange [2]. The Russian mathematitian Sofia Kowalevski (Kovalevskaya) [3] succeeded in 1888 to isolate the third general integrable case. Using a purely mathematical property, that explicit solution of the Euler–Poisson equations in each of the previous two cases is expressed in elliptic functions of time, Kowalevski set her aim to find all cases sharing the property that the solution has no singularities in the complex t-plane other than poles for arbitrary initial conditions of motion. As a result, she retrieved the two classical cases and one more: the case known now under her name. For the last case, Kowalevski succeeded to find the fourth integral and also expressed the solution explicitly in terms of ultra-elliptic functions of time.

After the success of Kowalevsi, a whole generation of Russian mathematitians turned to intensive work on problems of rigid body dynamics and, in particular, the classical problem of motion of a heavy body about a fixed point in a uniform gravity field. The list of names includes Joukovsky, Lyapunov, Steklov, Chaplygin, Goriachev and others. Their efforts produced so significant results that rendered the classical problem, almost exclusively, to a russian subject for a quite long period.

1. In his 1899 work [7], Goriachev obtained the condition (7), and the expressions (10) (provided below), but he erroneously concluded that the integral (11) is hyperelliptic.

2. In 1965 Kharlamov [8] derived a relation equivalent to (13) and concluded that the motion is periodic. He also evaluated the time period of the motion, but he didn't proceed to invert the elliptic integral.

3. In 1974 Stepanova [9] gave an explicit inversion formula of the integral (11) in the form

$$p = p_0 \sqrt{\frac{1 - \mathrm{sn}(\chi t, k_0)}{1 + n\,\mathrm{sn}(\chi t, k_0)}}$$

   and expressed the six Euler–Poisson variables as functions of time. Unfortunately, the solution given by Stepanova is not adequate for describing the motion, since it involves odd powers the auxiliary quantity p, which is not differentiable at all its zeros and takes the wrong sign on half of each period. As a result, expressions for p and r are not analytic functions of time. Moreover, expressions for the parameters m, n and the modulus k of the Jacobi elliptic functions are extremely complicated. The expression for γ_2 in [9] is written erroneously. A corrected expression for γ_2 is given in [6] (§ 8.3). The authors of [6] remarked also that Stepanova's solution is two-valued and that its period is only half the period found earlier by Kharlamov based on the analysis of the quadrature in (13), but they did not realize the need to modify the way of inversion of the integral used by Stepanova.

The aim of the present work is to provide a natural and analytic solution of Goriachev's case. As a result we were also able to give graphical illustration of the solution.

1. Conditions and solution. Suppose that the centre of mass lies on the first principal axis, i.e. $y_0 = z_0 = 0$. Equations of motion (1) take the form

$$\begin{aligned} A\dot{p} + (C - B)qr &= 0, \\ B\dot{q} + (A - C)\,pr &= a\gamma_3, \\ C\dot{r} + (B - A)pq &= -a\gamma_2, \end{aligned} \tag{3}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \quad \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \quad \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0, \tag{4}$$

where we have put $Mgx_0 = a$. Without loss of generality we assume $a > 0$. The general first integrals of motion become

$$\begin{aligned} I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + a\gamma_1 = h, \\ I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f, \\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \end{aligned} \tag{5}$$

where h, f are certain parameters, determined by initial state of motion.

For a solution Goriachev [7] tries two relations of the form

$$\gamma_2 = pq(n_1 + n_2 p^2), \qquad \gamma_3 = n_3 pr, \tag{6}$$

and finds that this is possible only when the principal moments of inertia of the body are subject to the restriction

$$A(9B - 8C) = 16C(B - C). \tag{7}$$

In the plane of inertia parameters the condition (7) is satisfied on the curve $PQNR$ (Fig. 1), so that the character of motion of a body is determined by the point representing the body on that curve. Denoting the ratio C/A by c and solving (7) for B one can write

$$C = Ac, \qquad B = A\frac{8c(2c - 1)}{16c - 9}. \tag{8}$$

Regarding triangle inequalities, the parameter c ranges from $\frac{3}{5}$ at P to ∞ at R (the point at infinity).

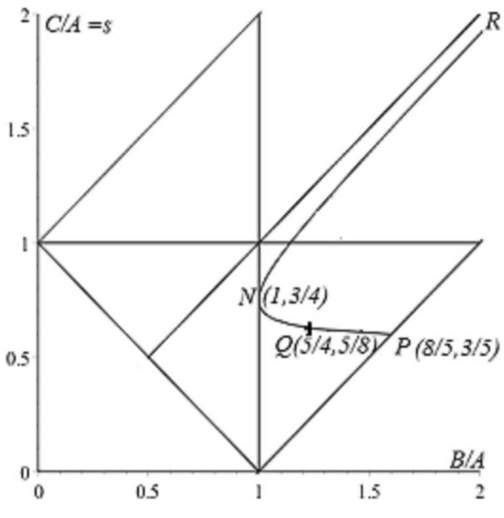

Fig. 1: Goriachev's condition (7) is satisfied on the hyperbolic branch PNQR.

From the equations of motion, it follows also that the areas and energy parameters of the motion must take the values

$$
\begin{aligned}
&f = 0,\\
&h = 2\varepsilon a\frac{(2c-1)(9-56c+64c^2)}{(3-4c)(3-8c)(3-4c)},
\end{aligned}
\tag{9}
$$

where

$$\varepsilon = \pm 1.$$

Under the above conditions five of the Euler–Poisson's variables can be expressed in terms of p as

$$
\begin{aligned}
q &= \frac{16c-9}{4}\sqrt{\frac{32\varepsilon(2c-1)a}{c(4c-3)(8c-5)(8c-3)A}-\frac{p^2}{c^2}},\\
r &= \frac{1}{8}\Big\{-\frac{128\varepsilon(16c-9)a}{c(4c-3)A}+32\frac{(4c-3)(48c^2-44c+9)}{c^2(2c-1)}p^2-\\
&\quad -\frac{\varepsilon(8c-5)(8c-3)(4c-1)(4c-3)^3A}{c^3(2c-1)^2a}p^4\Big\}^{1/2},\\
\gamma_1 &= \varepsilon-\frac{(4c-3)(16c^2-15c+3)A}{2c(2c-1)a}p^2-\\
&\quad -\frac{(8c-3)(8c-5)(4c-1)(4c-3)^3A^2}{128c^2(2c-1)^2a^2}p^4,\\
\gamma_2 &= \frac{(4c-3)(8c-3)(8c-5)A}{2(2c-1)(16c-9)a}pq\Big[-1+\varepsilon\frac{(4c-1)(4c-3)^2A}{16c(2c-1)a}p^2\Big],\\
\gamma_3 &= \frac{(3-4c)A}{2a}pr,
\end{aligned}
\tag{10}
$$

while p is determined as a function of time from the relation

$$\int^p \frac{dp}{\sqrt{F(p)}} = \frac{B-C}{A}t, \tag{11}$$

where $F(p) = q^2(p)r^2(p)$ is a polynomial of the sixth degree in p.

It remains to ensure that the two square roots figuring in the expressions for q and r in (10) take only real values, depending on the ratio c of the principal moments of inertia of the body and the sign of ε. Regarding the first expression in (10), the arc PR is devided into 3 parts:

1. PQ, on which $c \in i_0 = [\frac{3}{5}, \frac{5}{8})$. On this arc one should choose $\varepsilon = 1$ and then $h \in [\frac{26}{9}a, \infty)$.

2. QN, on which $c \in (\frac{5}{8}, \frac{3}{4})$. On this arc if one chooses $\varepsilon = -1$, q can take real values. However, in that case all the three coefficients under the square root sign in the expression for r become negative, so that r takes only imaginary values.

3. NR, with $c \in (\frac{3}{4}, \infty)$ and $\varepsilon = 1$. On this arc, it can be shown that $h < -a$, so that the kinetic energy of the body is negative.

Thus, the motion is possible only for bodies corresponding to points of the arc PQ with the choice $\varepsilon = 1$.

In fact, with the use of the change of variable

$$p = 4\sqrt{\frac{2c(2c-1)a}{(3-4c)(8c-3)(5-8c)A}v}\ , \tag{12}$$

the relation (11) is rendered to

$$2\sqrt{\frac{(4c-1)(3-4c)a}{c(8c-3)(5-8c)A}}t = \int \frac{dv}{\sqrt{(v-v_0)v(v_1-v)(1-v)}}\ , \tag{13}$$

where

$$v_{0,1} = -1/2\,\frac{48\,c^2 - 44\,c + 9}{(4\,c-1)\,(3-4c)} \mp 1/4\,\frac{\sqrt{2\,(9-64c+160\,c^2-128\,c^3)}}{(4\,c-1)\,\sqrt{(3-4\,c)}}. \tag{14}$$

We first note that, as shown in Figure 2, on the interval $i_0 = [\frac{3}{5}, \frac{5}{8})$, the roots of the 4th degree polynomial under the square root sign in (13) have the order $v_0 < 0 < v_1 < 1$.

The polynomial under the quadratic root sign in (13) has degree 4, and thus the integral is elliptic of the first kind. From tables of integrals, e.g. [10], we evaluate this integral and hence obtain the inversion formula

$$v = \frac{n\mathrm{sn}^2(u,k)}{1-m\mathrm{sn}^2(u,k)}\ , \tag{15}$$

where

$$
\begin{aligned}
&u = \mu t + u_0,\\
&\mu = \frac{\sqrt[4]{(4c-3)(128c^3-160c^2+64c-9)}\sqrt{a}}{\sqrt{2c(8c-3)(5-8c)A}},\\
&k = \sqrt{\frac{v_1(1-v_0)}{v_1-v_0}} = \sqrt{\frac{1}{2}+\frac{\sqrt{2}(256c^3-320c^2+144c-27)}{8(4c-3)^{3/2}\sqrt{128c^3-160c^2+64c-9}}},\\
&n = \frac{2^{3/2}}{16}\frac{(8c-5)(8c-3)(16c-9)}{\sqrt{(4c-3)(128c^3-160c^2+64c-9)}},\\
&m = \frac{1}{2}\Big[1-\frac{\sqrt{2}(48c^2-44c+9)}{\sqrt{(4c-3)(128c^3-160c^2+64c-9)}}\Big]
\end{aligned}
\tag{16}
$$

and u_0 is an arbitrary constant. Figure (3) shows the graph for $k(c)$ as drawn from the last expression. It ranges from 0.87064540 at $s = \frac{3}{5}$ to 0 at $c = \frac{5}{8}$.

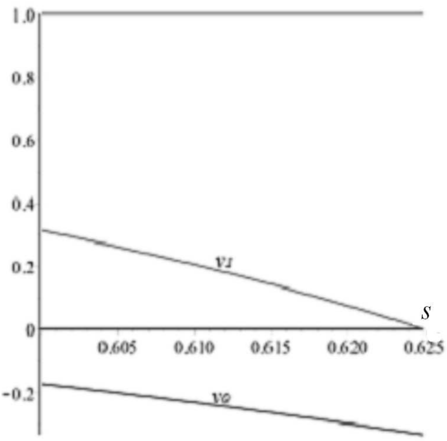

Fig. 2: The order of the roots of the 4th-degree polynomial in (13).

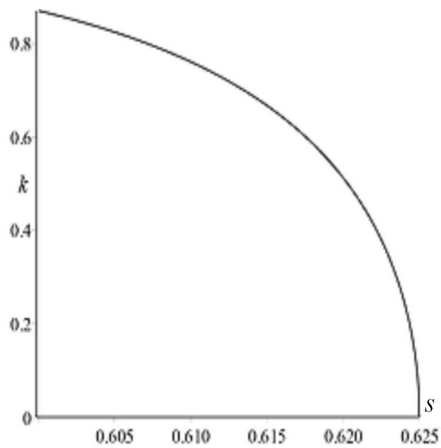

Fig. 3: k v. c.

Using (15) we express the components of the angular velocity as:

$$
\begin{aligned}
p &= p_1\sqrt{\frac{a}{A}}\frac{\operatorname{sn}(u,k)}{\sqrt{1-m\operatorname{sn}^2(u,k)}},\\
q &= q_1\sqrt{\frac{a}{A}}\frac{\operatorname{dn}(u,k)}{\sqrt{1-m\operatorname{sn}^2(u,k)}},\\
r &= r_1\sqrt{\frac{a}{A}}\frac{\operatorname{cn}(u,k)}{1-m\operatorname{sn}^2(u,k)},
\end{aligned}
\tag{17}
$$

in which

$$p_1 = \frac{2\sqrt{\sqrt{2}c(2c-1)(16c-9)}}{\sqrt[4]{(4c-3)(128c^3-160c^2+64c-9)}},$$
$$q_1 = (16c-9)\sqrt{\frac{2(2c-1)}{c(4c-3)(8c-3)(8c-5)}}, \tag{18}$$
$$r_1 = \sqrt{\frac{2(16c-9)}{c(3-4c)}}.$$

The components of the vertical unit vector γ take the form:

$$\gamma_1 = 1 - 2\sqrt{2}\frac{(16c-9)(16c^2-15c+3)}{\sqrt{(4c-3)^3(128c^3-160c^2+64c-9)}}\frac{\mathrm{sn}^2(u,k)}{1-m\mathrm{sn}^2(u,k)} -$$
$$-\frac{(4c-1)(8c-3)(5-8c)(16c-9)^2}{4(4c-3)^2(128c^3-160c^2+64c-9)}\frac{\mathrm{sn}^4(u,k)}{(1-m\mathrm{sn}^2(u,k))^2},$$
$$\gamma_2 = \frac{(8c-3)(8c-5)(4c-3)}{8(2c-1)(16c-9)}p_1q_1\frac{\mathrm{sn}(u,k)\mathrm{dn}(u,k)}{(1-m\mathrm{sn}^2(u,k))}\times \tag{19}$$
$$\times[\frac{\sqrt{2}(4c-1)(16c-9)}{\sqrt{(4c-3)(128c^3-160c^2+64c-9)}}\frac{\mathrm{sn}^2(u,k)}{1-m\mathrm{sn}^2(u,k)} - 4],$$
$$\gamma_3 = \frac{3-4c}{2}p_1r_1\frac{\mathrm{sn}(u,k)\mathrm{cn}(u,k)}{(1-m\mathrm{sn}^2(u,k))^{3/2}}.$$

The formulas (17)–(19) describe analytical functions on the whole time line. All those expressions, as well as coefficients in 16, are incomparably simpler than the ones given in [9].

2. Properties of the motion.

2.1. The initial motion. To obtain symmetric views in the graphics we give the constant u_0 the value $K(k)$. This corresponds the choice of the initial moment $t=0$ as the one at which $p=0$. At this same moment, from (19), we have $\gamma = (1,0,0)$. The x-axis carrying the center of mass begins from a position vertically above the fixed point. The initial angular velocity that should be given to the body to commence a motion of the Goriachev type lies in the yz-plane and its direction depends on the parameter c. Figure 4-a shows the variation of the dimentionless quantities $\left(q^0 = \sqrt{\frac{A}{a}}q,\ \ r^0 = \sqrt{\frac{A}{a}}r\right)$ as c varies on the interval i_0. From this figure, it is obvious that as c tends to $\frac{5}{8}$, q tends to infinity while r tends to a finite limit $\frac{4\sqrt{10}}{5}\sqrt{\frac{a}{A}} \simeq 2.5298\sqrt{\frac{a}{A}}$.

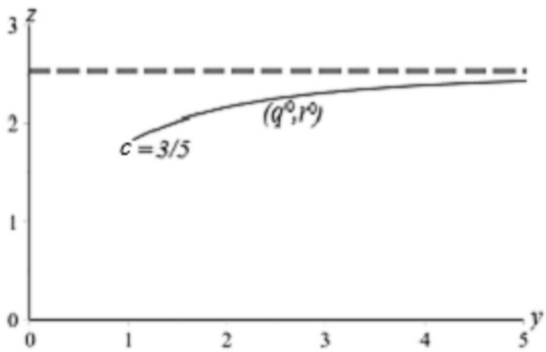

Fig. 4-a: *The initial angular velocity* (q^0, r^0) *as c varies.*

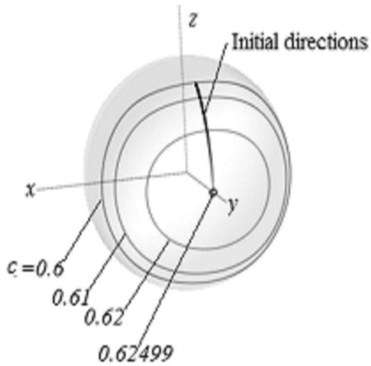

Fig. 4-b: *Directions of angular velocity during motion as c varies.*

2.2. Periodicity of the motion. We now clarify the general character of motion after this initial setting. We readily note that the second component q of the angular velocity does not change sign. In a time period $\dfrac{2K(k)}{\mu}$, it ranges from a minimum q_0 to a maximum $\dfrac{q_0}{\sqrt{1-v_1}}$ and then to q_0 again. The motion of the body is composed of a rotation around the y-axis and vibrations around the x, z-axes. The whole motion is periodic with period

$$T = \frac{4K(k)}{\mu} = 4K(k)\frac{\sqrt{2c(8c-3)(5-8c)A}}{\sqrt[4]{(3-4c)(9-64c+160c^2-128c^3)}\sqrt{a}}. \tag{20}$$

The period of motion decreases monotonically from its value at $c = 3/5$ to zero at $c = 5/8$, corresponding to very fast uniform rotation about the y-axis.

In Fig. 4-b we show 4 closed curves representing the traces on the unit sphere fixed in the body by the unit vector $\dfrac{\omega}{\omega}$ in the direction of the angular velocity along a period of the motion. All curves are closed around the y-axis. As c approaches 5/8, the curve becomes a very small one shrinking to that axis.

2.3. Orbits of motion on the Poisson sphere. Now, we turn to the picture of the trajectories of the motion, the trace of the vertical vector γ on the Poisson sphere. As we have seen above, the point $P(1, 0, 0)$ is common between all orbits. Figure 5 shows the space view of the orbit corresponding to the value $c = 3/5$, as an example of the generic orbits. It begins from the point P on the x-axis and goes through the points $P_1, Q, R, P_2, P, P_3, Q, R, P_4$ and then closes at P. The orbit has three self-intersection points P, Q and R, so that it makes three loops. The direction of motion is shown on all arcs of the orbit on the two halves of an opaque sphere (Figure 6a, 6b).

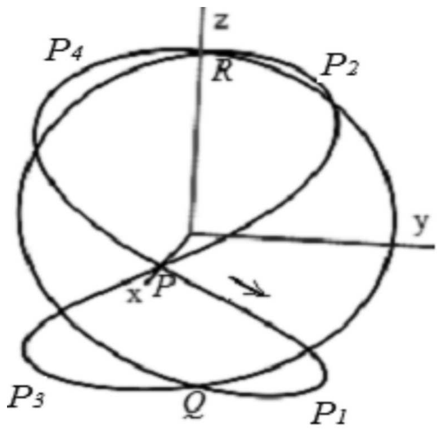

Fig. 5: *Space view of the trajectory for* $c = 0.6$ *on a transparent Poisson sphere.*

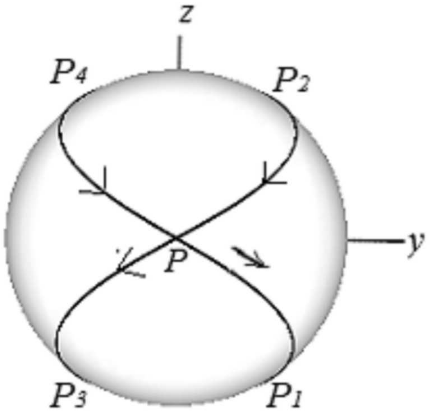

Fig. 6-a:

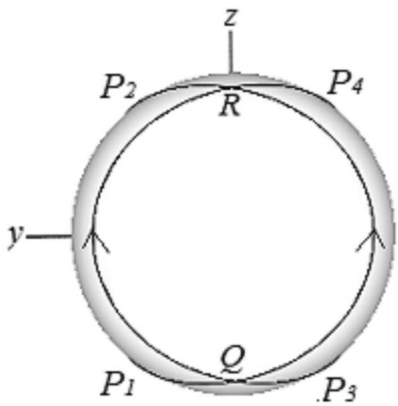

Fig. 6-b:

Although those orbits are not simple in the space view, they have two planes of symmetry: xz and xy. The projections of four orbits on the xz-plane are shown in Figure 7 for values of c ranging from the beginning to near the end of the interval $[\frac{3}{5}, \frac{5}{8})$. Each orbit can be seen as the intersection of the Poisson sphere with a cylinderical surface, which is parallel to the y-axis and touches the unit sphere at three points. With increasing c the projection becomes wider and encloses the orbits with smaller values of c. As $c \to 5/8$, the projection of the orbit approaches a circle of unit radius. In fact, the limiting orbit at this value of c is the circle $\gamma_2 = 0$, corresponding to very fast rotations about the y-axis, which takes a horizontal position.

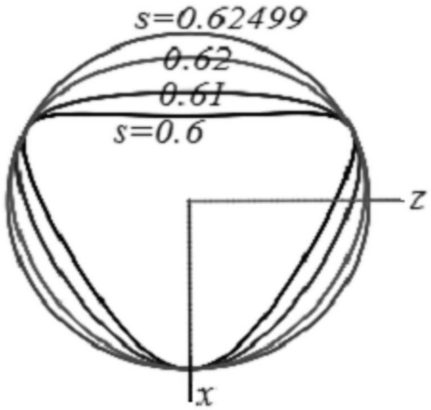

Fig. 7: Projections of trajectories *on the xz-plane corresponding to four values of c. (View from the top of −y-axis)*

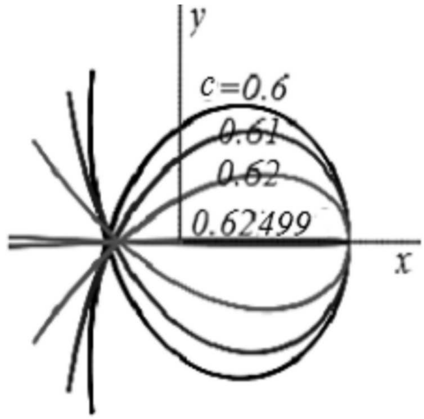

Fig. 8: Projections of trajectories *on the xy-plane corresponding to four values of c. (View from the top of z-axis)*

Figure 8 depicts the projections of the same orbits on the xy-plane. This projection begins wider at $c = 3/5$ and becomes narrower with increasing c to coincide in the limit, as $c \to 5/8$, with the diameter of the sphere on the x-axis. Figure 9 shows the same orbits on the front and rear halves of the sphere.

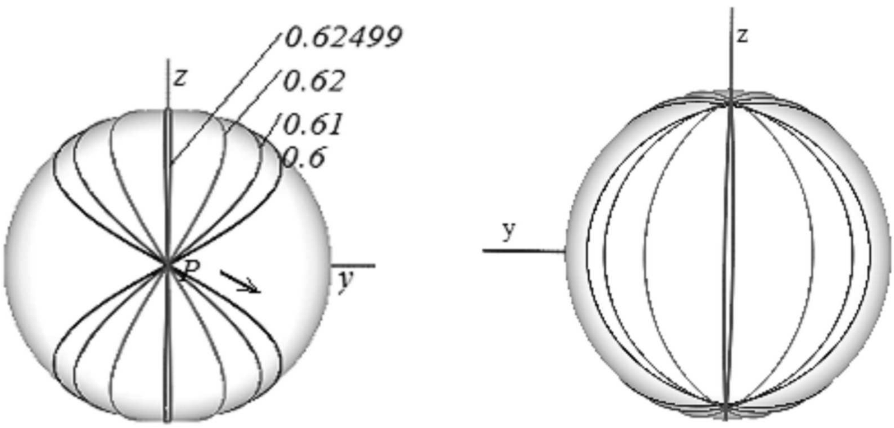

Fig. 9: Front and rear views of the four trajectories on the Poisson sphere.

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Х.М. Яхья

Решение Горячева в динамике твердого тела с неподвижной точкой

Построены естественная и аналитическая параметризация выражений для переменных Эйлера–Пуассона в случае, найденном Горячевым в 1899 г. Решение, данное Степановой в 1974 г., не пригодно для описания движения на всем промежутке времени.

Ключевые слова: *решение Горячева, эллиптические функции, твердое тело.*

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