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On the Partial Equiasymptotic Stability in Functional Differential Equations

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A system of functional differential equations with delay $dz/dt = Z(t, z_t)$, where Z is the vector-valued functional is considered. It is supposed that this system has a zero solution $z = 0$. Definitions of its partial stability, partial asymptotical stability, and partial equiasymptotical stability are given. Theorems on the partial equiasymptotical stability are formulated and proved.

Key Words: functional differential equations; Lyapunov functionals; equiasymptotic stability.

1. INTRODUCTION

Let $t \in R_+ = [0; \infty)$, $x = (x^1, \ldots, x^n) \in R^n$, $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$, $y = (y^1, \ldots, y^m)$, $|y| = \sqrt{(y^1)^2 + \cdots + (y^m)^2}$, $z = (x; y) = (z^1, \ldots, z^{n+m}) \in R^{n+m}$. For a given $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^{n+m}. Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n+m}) = (\psi; \lambda) \in C$, where $\psi = (\psi_1, \ldots, \psi_n)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$. Denote

$$\|\psi\| = \sup(|\psi_i(\theta)| \text{ under } -h \le \theta \le 0,\ 1 \le i \le n),$$

$$\|\lambda\| = \sup(|\lambda_j(\theta)| \text{ under } -h \le \theta \le 0,\ 1 \le j \le m),$$

$$\|\varphi\| = \max(\|\psi\|, \|\lambda\|),$$

$$B_H = \{\varphi \in C : \|\varphi\| \le H\},$$

$$C_H = \{\varphi \in C : \|\psi\| \le H, \|\lambda\| < +\infty\}.$$

If z is a continuous function of u defined on $-h \le u < A$, $A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then z_t denotes the restriction of z

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to the segment $[t-h, t]$ so that $z_t = (z_t^1, \ldots, z_t^{n+m}) = (x_t; y_t)$ is an element of C defined by $z_t(\theta) = z(t+\theta)$ for $-h \le \theta \le 0$.

Consider a system of functional differential equations

$$\frac{dz}{dt} = Z(t, z_t). \tag{1.1}$$

In this system dz/dt denotes the right-hand derivative of z at t, t is time, and $Z(t, \varphi) = (X(t, \varphi), Y(t, \varphi)) \in R^{n+m}$ is defined on $R_+ \times C_H$; $X \in R^n$, $Y \in R^m$, $Z(t; 0) \equiv 0$.

According to T. Burton [4], we denote by $z(t_0, \varphi) = (x(t_0, \varphi), y(t_0, \varphi))$ a solution of (1.1) with initial condition $\varphi \in C_H$, where $z_{t_0}(t_0, \varphi) = \varphi$ and we denote by $z(t, t_0, \varphi)$ the value of $z(t_0, \varphi)$ at t and $z_t(t_0, \varphi) = z(t+\theta, t_0, \varphi)$, $-h \le \theta \le 0$.

It is assumed that the vector-valued functional $Z(t, \varphi)$ is continuous on $[0; \infty) \times C_H$ so that a solution will exist for each continuous initial condition. We suppose that each solution $z(t_0, \varphi)$ is defined for those $t \ge t_0$ that $\|x_t(t_0, \varphi)\| < H$.

Let $V(t, \varphi)$ be a continuous functional defined for $t \ge 0$, $\varphi \in C_H$. The upper right-hand derivative of V along solutions of (1.1) is defined to be [4, 8, 10, 11]

$$\begin{aligned} \dot{V}(t, z_t(t_0, \varphi)) &= \frac{dV(t, z_t(t_0, \varphi))}{dt} \\ &= \overline{\lim_{\Delta t \to +0}} \{V(t+\Delta t, z_{t+\Delta t}(t_0, \varphi)) - V(t, z_t(t_0, \varphi))\} \frac{1}{\Delta t}. \end{aligned}$$

If V satisfies a Lipschitz condition in the second argument, then this limit is uniquely determined.

In [15, 16, 18] the partial stability results were obtained for ordinary differential equations. The goal of this paper is to prove analogous results for functional differential equations (1.1).

2. DEFINITIONS AND PRELIMINARY RESULTS

DEFINITION 2.1. The trivial solution

$$z(t) \equiv 0 \tag{2.1}$$

of system (1.1) is called partially stable with respect to x (x-stable) if for every $\varepsilon > 0$ and $t_0 \in R_+$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that inequality $|x(t, t_0, \varphi)| < \varepsilon$ holds for $t \ge t_0$, if $\varphi \in B_\delta$.

DEFINITION 2.2. If δ does not depend on t_0 in Definition 2.1 (i.e., $\delta = \delta(\varepsilon)$), then solution (2.1) is called partially uniformly stable with respect to x (or uniformly x-stable).

We shall consider various kinds of attraction by analogy to ordinary differential equations [13].

DEFINITION 2.3. Solution (2.1) of Eqs. (1.1) is called partially attractive with respect to x (or x-attractive), if for every $t_0 \in R_+$ there exists $\eta = \eta(t_0) > 0$ and for every $\varepsilon > 0$ and $\varphi \in B_\eta$ there exists $\sigma = \sigma(\varepsilon, t_0, \varphi) > 0$ such that $|x(t, t_0, \varphi)| < \varepsilon$ for any $t \geq t_0 + \sigma$. In this case we shall say that the domain of x-attraction at t_0 contains the set B_η.

DEFINITION 2.4. Solution (2.1) of system (1.1) is called x-equiattractive (or equiattractive with respect to variable x), if for every $t_0 \geq 0$ there exists $\eta = \eta(t_0) > 0$, and for any $\varepsilon > 0$ there is $\sigma = \sigma(\varepsilon, t_0) > 0$ such that $|x(t, t_0, \varphi)| < \varepsilon$ for all $\varphi \in B_\eta$ and $t \geq t_0 + \sigma$.

DEFINITION 2.5. The zero solution of Eqs. (1.1) is called uniformly x-attractive if for some $\eta > 0$ and any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that $|x(t, t_0, \varphi)| < \varepsilon$ for all $\varphi \in B_\eta$, $t_0 \geq 0$, and $t \geq t_0 + \sigma$.

DEFINITION 2.6. The trivial solution (2.1) of system (1.1) is called:

—asymptotically x-stable if it is x-stable and x-attractive;

—equiasymptotically x-stable (or partially equiasymptotically stable with respect to the variable x) if it is x-stable and x-equiattractive;

—uniformly asymptotically x-stable if it is uniformly x-stable and uniformly x-attractive.

DEFINITION 2.7. A functional $W(\psi)$, independent on t, is called x-positive definite, if $W(\psi) \geq 0$, and also $W(\psi) = 0$ iff $\|\psi\| = 0$. A functional $V(t, \varphi)$ is called x-positive definite, if there exists x-positive definite functional $W(\psi)$ such that $V(t, \varphi) \geq W(\psi)$, $V(t, 0) \equiv 0$. A functional $V(t, \varphi)$ is called x-negative definite, if $-V(t, \varphi)$ is an x-positive one.

By analogy to ordinary differential equations [13, 15], one can show that $V(t, \varphi)$ is x-positive definite iff there exists a function $a \in K$ such that $V(t, \varphi) \geq a(\|\psi\|)$. Here K is the class of Hahn functions [7, 13].

DEFINITION 2.8. A solution $z(t_0, \varphi)$ of functional differential equations (1.1) is called y-bounded if $|x(t, t_0, \varphi)| < \zeta < H$ for $t \geq t_0$ implies that there exists a number $N_\zeta > 0$ such that $|y(t, t_0, \varphi)| < N_\zeta$ for $t \geq t_0$.

Consider some sufficient conditions of partial equiasymptotic stability.

THEOREM 2.1. *Let the right-hand side of system* (1.1) *be bounded on* $R_+ \times C_H$, *and any solution* $z(t_0, \varphi)$ *be* y*-bounded. If a continuous functional* $V(t, \varphi)$, *such that* $V(t, 0) \equiv 0$, *satisfies the condition*

$$V(t, \varphi) \geq a(|\psi(0)|), \quad a \in K, \tag{2.2}$$

and for every $t_0 \geq 0$ *there exists* $q(t_0) > 0$ *such that* $\varphi \in B_q$ *implies that* $V(t, z_t(t_0, \varphi))$ *does not increase monotonically and tends to zero as* $t \to +\infty$, *then solution* (2.1) *of system* (1.1) *is equiasymptotically x-stable.*

Proof. The conditions of the theorem imply the x-stability of solution (2.1) [18]. Let us prove its x-equiattraction. Let $t_0 \geq 0$ be an arbitrary initial moment of time, and $0 < \zeta < H$. Choose some positive η, satisfying the condition $|x(t, t_0, \varphi)| < \zeta < H$ if $\varphi \in B_\eta$ and $\eta = \eta(t_0) < q(t_0)$. For any $t_0 \geq 0$, $\varepsilon > 0$, $\varphi \in B_\eta$ there exists $T = T(\varepsilon, t_0, \varphi) > 0$ such that

$$V(t_0 + T, z_{t_0+T}(t_0, \varphi)) < \frac{1}{2} a^{-1}(\varepsilon),$$

where a^{-1} is the function, inverse to the function a. The solution $z(t_0, \varphi)$ continuously depends on initial data, and the functional $V(t, \varphi)$ is continuous in its arguments. Hence, there is a neighborhood $Q(\varphi)$ of the function φ in B_η such that for each $\varphi_0 \in Q(\varphi)$ the inequality $V(t_0 + T, z_{t_0+T}(t_0, \varphi)) < a^{-1}(\varepsilon)$ is valid. Since V does not increase along solutions of system (1.1), then

$$V(t, z_t(t_0, \varphi_0)) < a^{-1}(\varepsilon) \text{ for any } t \geq t_0 + T(\varepsilon, t_0, \varphi_0), \quad \varphi_0 \in Q(\varphi).$$

From the choice of the number η one can infer that $|x(t, t_0, \varphi_0)| < \zeta < H$, $|y(t, t_0, \varphi_0)| < N_\zeta$, and from the boundedness of $Z(t, \varphi)$ it follows that the set of functions $\{z_t(t_0, \varphi_0)\}$ $(t \geq t_0 + h, \varphi_0 \in B_\eta)$ is the family of uniformly bounded and equicontinuous functions [11]; i.e., this set is a compact one. Thus, the compact set of functions is covered by the class of neighborhoods $Q(\varphi)$. Hence, by the Heine–Borel theorem [14], there exists a finite subcovering $Q_1, Q_2, \ldots, Q_k$ of this covering with corresponding numbers

$$T_1 = T(\varepsilon, t_0, \varphi_1), \quad T_2 = T(\varepsilon, t_0, \varphi_2), \ldots, \quad T_k = T(\varepsilon, t_0, \varphi_k),$$

where $\varphi_i \in B_\eta$ $(i = 1, \ldots, k)$ are some fixed functions. Denote $\sigma(t_0, \varepsilon) = \max\{t_0 + h, T_1, T_2, \ldots, T_k\}$. Then $V(t, z_t(t_0, \varphi)) < a^{-1}(\varepsilon)$ for any $\varphi \in B_\eta$, $t \geq t_0 + \sigma(\varepsilon, t_0)$. This relation and the inequality (2.2) imply

$$|x(t, t_0, \varphi)| < \varepsilon \text{ under } t \geq t_0 + \sigma(\varepsilon, t_0).$$

This completes the proof.

THEOREM 2.2. *Let system* (1.1) *be such that*

(1) *there exists a functional* $V(t, \varphi)$, *satisfying inequality* (2.2), *and* $V(t, 0) \equiv 0$,

(2) $dV/dt \leq 0$,

(3) *for any* $\zeta > 0$, *inequalities* $V(t, z_t) > \zeta$, $\|x_t\| < H$ *imply*

$$\frac{dV(t, z_t)}{dt} \leq -m_\zeta(t), \tag{2.3}$$

$$\int_{t_0}^{\infty} m_\zeta(t)\, dt = +\infty. \tag{2.4}$$

Then solution (2.1) *of Eqs*. (1.1) *is equiasymptotically x-stable*.

Proof. From the conditions of the theorem it follows that for any $t_0 \geq 0$, $\varepsilon > 0$ there exists such $\delta = \delta(\varepsilon, t_0) > 0$ that $\varphi \in B_\delta$ implies $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$. Let us show that $V(t, z_t(t_0, \varphi))$ is a monotone nonincreasing function, and

$$\lim_{t\to\infty} V(t, z_t(t_0, \varphi)) = 0 \text{ for any } \varphi \in B_\delta. \tag{2.5}$$

The condition $dV/dt \leq 0$ implies a lack of increase of $V(t, z_t(t_0, \varphi))$. Let us prove relation (2.5). Suppose that this is not true; i.e., there exists $\zeta > 0$ such that $V(t, z_t(t_0, \varphi)) \geq \zeta$. The inequalities

$$V(t, z_t(t_0, \varphi)) \leq V(t_0, \varphi) + \int_{t_0}^{t} \frac{dV(\tau, z_\tau(t_0, \varphi))}{d\tau} d\tau$$

and (2.3) imply

$$0 \leq V(t, z_t(t_0, \varphi)) \leq V(t_0, \varphi) - \int_{t_0}^{t} m_\zeta(\tau)\, d\tau.$$

But this inequality is not true for t large enough because of condition (2.4). The contradiction proves relation (2.5). In view of Theorem 2.1 we conclude solution (2.1) of system (1.1) to be equiasymptotically x-stable.

THEOREM 2.3. *If the functional* $V(t, \varphi)$ *is such that* $V(t, 0) \equiv 0$,

$$V(t, \varphi) \geq \xi(t) a(|\psi(0)|), \quad a \in K, \tag{2.6}$$

where $\xi(t)$ *is a monotonically increasing function such that* $\xi(0) = 1$, $\lim_{t\to+\infty} \xi(t) = +\infty$, *and* $dV/dt \leq 0$, *then solution* (2.1) *of system* (1.1) *is equiasymptotically x-stable*.

Proof. Pick any $\varepsilon_1 \in (0, H)$. From the partial stability of the zero solution of Eqs. (1.1) it follows that for every $t_0 \geq 0$ there exists $\delta = \delta(t_0) > 0$ such that for any $\varphi \in B_\delta$, we have $|x(t, t_0, \varphi)| < \varepsilon_1$ for $t \geq t_0$. Denote

$$\mu(t_0) = \sup_{\varphi\in B_\delta} V(t_0, \varphi).$$

From inequalities $dV/dt \leq 0$ and (2.6) we derive

$$a(|x(t, t_0, \varphi)|) \leq \frac{V(t, z_t(t_0, \varphi))}{\xi(t)} \leq \frac{V(t_0, \varphi)}{\xi(t)}. \tag{2.7}$$

For any positive ε there exists $\sigma = \sigma(\varepsilon, t_0) > 0$ such that $\xi(t) > \frac{\mu(t_0)}{a(\varepsilon)}$ for all $t \geq t_0 + \sigma$. Hence, from inequalities (2.7) we get $a(|x(t, t_0, \varphi)|)) < a(\varepsilon)$; therefore, $|x(t, t_0, \varphi)| < \varepsilon$ for all $\varphi \in B_\delta$, $t \geq t_0 + \sigma(\varepsilon, t_0)$. This completes the proof.

EXAMPLE 2.1. Consider the system of functional differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= y(t)\sin(x(t-h)+y(t-h)) - \frac{x(t)}{2(t+h+1)}, \\ \frac{dy(t)}{dt} &= -x(t)\sin(x(t-h)+y(t-h)), \end{aligned} \tag{2.8}$$

which has a zero solution. Let

$$V = \frac{1}{2}(x^2(t)+y^2(t)) + (t+h+1)x^2(t) \geq \xi(t)a(|x(t)|),$$

where $\xi(t) = (t+h+1)/(h+1)$, $a(|x(t)|) = (h+1)|x(t)|^2$. Then $dV/dt \equiv 0$. Hence, by Theorem 2.3, the zero solution of (2.8) is equiasymptotically x-stable.

3. PARTIAL EQUIASYMPTOTIC STABILITY IN ALMOST PERIODIC SYSTEMS

DEFINITION 3.1 [1–3, 5, 6, 12, 17, 19]. A continuous function $F(t)\colon R \to R^{n+m}$ is called almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l]$, $\alpha \in R$, contains at least one number τ such that $|F(t+\tau) - F(t)| < \varepsilon$ for every $t \in R$. A number τ is called an ε-almost period of F.

DEFINITION 3.2 [9]. A continuous functional $F(t, \varphi)\colon R \times C_r \to R^{n+m}$ $(0 < r < \infty)$ is called uniformly almost periodic in t if for every $\varepsilon > 0$ there exists $l = l(\varepsilon, r) > 0$ such that any segment $[\alpha, \alpha + l]$, $\alpha \in R$, contains at least one number τ such that $|F(t+\tau, \varphi) - F(t, \varphi)| < \varepsilon$ for every $t \in R$, $\varphi \in C_r$.

Remark. A continuous function $F(t)$, which satisfies Definition 3.1 is called uniformly almost periodic in [2, 3], so Definitions 3.1 and 3.2 are somewhat different from their corresponding definitions in [2, 3].

LEMMA 3.1 [9]. *Let $F_1(t), \ldots, F_N(t)\colon R \to R^{n+m}$ be almost periodic functions. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that any segment $[\alpha, \alpha + l]$, $\alpha \in R$, contains a number τ such that*

$$|F_i(t+\tau) - F_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, N; \quad t \in R.$$

We denote

$$C_{H(L)} = \{\varphi \in C_H : |\varphi(\theta_1) - \varphi(\theta_2)| \leq L|\theta_1 - \theta_2| \text{ for each } \theta_1, \theta_2 \in [-h, 0]\} \subset C_H.$$

LEMMA 3.2 [9]. *If the functional $F(t, \varphi)$: $R \times C_{H(L)} \to R^{n+m}$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_{H(L)}$, then it is uniformly almost periodic in t.*

We consider the system of functional differential equations (1.1) under the assumptions above. We also assume that the functional $Z(t, \varphi)$ is Lipschitzian in φ and almost periodic in t for every fixed $\varphi \in C_H$.

LEMMA 3.3 [9]. *Consider the solution $z(t_0, \varphi_0)$ of system* (1.1). *We suppose that $z_t(t_0, \varphi_0)$ belongs to C_r $(0 < r < H)$ for $t \geq 0$. Let $\{\varepsilon_k\}$ be a monotonically approaching zero sequence of positive numbers and $\{\tau_k\}$ a sequence of ε_k-almost periods of $Z(t, \varphi)$ (for every ε_k there corresponds an ε_k-almost period τ_k). Then the limit relation*

$$\lim_{k\to\infty} \|z_{t^*}(t_0, \varphi_k) - z_{t^*+\tau_k}(t_0, \varphi_0)\| = 0 \tag{3.1}$$

holds, where $\varphi_k = z_{t_0+\tau_k}(t_0, \varphi_0)$ and t^* is a fixed moment of time which is more than t_0 $(t^* > t_0)$.

THEOREM 3.1. *Let functional differential equations* (1.1) *satisfy the above conditions; let any solution $z(t_0, \varphi)$ be y-bounded, and there exists a continuous functional $V(t, \varphi)$: $R \times C_H \to R$, which is locally Lipschitz in φ, such that the following conditions are fulfilled on the set $R \times C_H$:*

(i) $V(t, 0) \equiv 0, \quad a(|\psi(0)|) \leq V(t, \varphi)$, *where $a \in K$;*

(ii) *$V(t, \varphi)$ is almost periodic in t for each fixed $\varphi \in C_H$;*

(iii) *$dV/dt \leq 0$, $dV/dt \not\equiv 0$ on each solution of system* (1.1).

Then the solution

$$z = 0 \tag{3.2}$$

of functional differential equations (1.1) *is equiasymptotically x-stable.*

Proof. From conditions (i) and (iii) it follows that solution (3.2) is x-stable [18]. Let $\varepsilon \in (0, H)$ be any positive number. Denote by $t_0 \in R$ the initial moment of time. By the x-stability of the zero solution there exists $\delta > 0$ such that if $\varphi \in B_\delta$, then $z_t(t_0, \varphi) \in C_\varepsilon$ for every $t \geq t_0$. Choose such a $\delta > 0$ and show that any solution $z(t_0, \varphi)$ with $\varphi \in B_\delta$ is x-equiattractive.

The function $V(t) = V(t, z_t(t_0, \varphi))$ is monotonically nonincreasing because $dV/dt \leq 0$. Hence there exists the limit

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V(t, z_t(t_0, \varphi)) = V_0,$$

and it is easy to see that $V(t, z_t(t_0, \varphi)) \geq V_0 \geq 0$ for $t \in [t_0, \infty)$. Let us show that $V_0 = 0$. Suppose that this is not true; i.e., assume that $V_0 > 0$.

Consider some monotonically approaching zero sequence $\{\varepsilon_k\}$ of positive numbers, where ε_1 is sufficiently small. By Lemma 3.2 for every ε_i there exists a sequence of ε_i-almost periods $\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,n}, \ldots \to \infty$ for functionals $Z(t, \varphi)$ and $V(t, \varphi)$ that inequalities

$$|V(t + \tau_{i,n}, \varphi) - V(t, \varphi)| < \varepsilon_i,$$

$$|Z(t + \tau_{i,n}, \varphi) - Z(t, \varphi)| < \varepsilon_i$$

hold for each $t \in R$, $\varphi \in C_{H(L)}$. Note that, if t is large enough, then $z_t \in C_{H(L)}$ [11]. Without loss of generality one can suppose $\tau_{i,n} < \tau_{i+1,n}$ for every i, n. Designate $\tau_k = \tau_{k,k}$.

Consider the sequence of functions $\varphi_k = z_{t_0+\tau_k}(t_0, \varphi)$ $(k = 1, 2, \ldots)$. It is a bounded sequence of equicontinuous functions because $\varphi_k \in C_\varepsilon$, $|y(t, t_0, \varphi)| < N_\varepsilon$; therefore there is a limit function φ^* of this sequence. Without loss of generality one can assume the sequence φ_k itself converges to φ^*. Because of continuity and almost periodicity of the functional $V(t, \varphi)$ we obtain

$$\begin{aligned} V(t_0, \varphi^*) &= \lim_{n\to\infty} V(t_0, \varphi_n) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} V(t_0 + \tau_k, \varphi_n) \\ &= \lim_{n\to\infty} V(t_0 + \tau_n, \varphi_n) \\ &= \lim_{n\to\infty} V(t_0 + \tau_n, z_{t_0+\tau_n}(t_0, \varphi_0)) = V_0. \end{aligned}$$

Now consider the solution $z(t_0, \varphi^*)$. From condition (iii) of the theorem, the existence of such moment of time t^* $(t^* > t_0)$ follows when the inequality

$$V(t^*, z_{t^*}(t_0, \varphi^*)) = V_1 < V_0$$

holds.

Solutions of functional differential equations (1.1) are continuous in initial data, so one can write

$$\lim_{k\to\infty} \|z_{t^*}(t_0, \varphi_k) - z_{t^*}(t_0, \varphi^*)\| = 0$$

because

$$\lim_{k\to\infty} \|\varphi_k - \varphi^*\| = 0.$$

Hence it follows

$$\lim_{k\to\infty} V(t^*, z_{t^*}(t_0, \varphi_k)) = V_1. \tag{3.3}$$

Using the uniform almost periodicity property of $Z(t, \varphi)$ and limit relation (3.1), we obtain the inequality

$$\|z_{t^*}(t_0, \varphi_k) - z_{t^*+\tau_k}(t_0, \varphi_0)\| \leq \gamma_k, \tag{3.4}$$

where $\gamma_k \to 0$ as $k \to \infty$. Because of uniform almost periodicity property of $V(t, \varphi)$ we have

$$|V(t^*, \varphi) - V(t^* + \tau_k, \varphi)| < \varepsilon_k \tag{3.5}$$

for every $\varphi \in C_H$ and from conditions (3.3) and (3.4) it follows that

$$|V(t^*, z_{t^*+\tau_k}(t_0, \varphi)) - V_1| < \eta_k, \tag{3.6}$$

where $\eta_k \to 0$ as $k \to \infty$.

From (3.5) we obtain

$$|V(t^*, z_{t^*+\tau_k}(t_0, \varphi)) - V(t^* + \tau_k, z_{t^*+\tau_k}(t_0, \varphi))| < \varepsilon_k. \tag{3.7}$$

From (3.6) and (3.7) we have

$$|V(t^* + \tau_k, z_{t^*+\tau_k}(t_0, \varphi)) - V_1| < \eta_k + \varepsilon_k, \tag{3.8}$$

where $\eta_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand

$$\lim_{k\to\infty} V(t^* + \tau_k, z_{t^*+\tau_k}(t_0, \varphi)) = V_0. \tag{3.9}$$

The relations (3.8) and (3.9) are in contradiction to the inequality $V_1 < V_0$. The obtained contradiction proves that $V_0 = 0$.

Thus, we have proved that for any $t_0 \in [0, \infty)$ there exists a $\delta = \delta(t_0) > 0$, such that $\varphi \in B_\delta$ implies that $V(t, z_t(t_0, \varphi))$ is monotonically nonincreasing and $\lim_{t\to\infty} V(t, z_t(t_0, \varphi)) = 0$. Hence, from Theorem 2.1 it follows that solution (3.2) of functional differential equations (1.1) is equiasymptotically x-stable. The proof is complete.

EXAMPLE 3.1. Consider the nonlinear system of functional differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= -f(t, x_t, y_t)y(t) + (\sin^2 t + \sin^2 \pi t - 4)x^3(t) + 4x^2(t)x(t-h) \\ &\quad -6x(t)x^2(t-h) + 4x^3(t-h), \\ \frac{dy(t)}{dt} &= f(t, x_t, y_t)x(t), \end{aligned} \tag{3.10}$$

where $f(t, \psi, \lambda)$ is a continuous bounded functional, which is almost periodic in t for any fixed functions ψ and λ. This system has a zero solution

$$x(t) \equiv 0, \quad y(t) \equiv 0. \tag{3.11}$$

The derivative of the x-positive definite functional

$$V = \frac{1}{2}(x^2(t) + y^2(t)) + \int_{t-h}^{t} x^4(s)\,ds$$

along the solutions of (3.10) is

$$\begin{aligned}\frac{dV}{dt} &= (\sin^2 t + \sin^2 \pi t - 4)x^4(t) + 4x^3(t)x(t-h) - 6x^2(t)x^2(t-h)\\ &\quad + 4x(t)x^3(t-h) + x^4(t) - x^4(t-h)\\ &= -[x(t) - x(t-h)]^2 + (\sin^2 t + \sin^2 \pi t - 2)x^4(t).\end{aligned}$$

This derivative is not negative definite, but it is negative for any fixed t for every nonzero solution of Eqs. (3.10). Therefore, by Theorem 3.1, zero solution (3.11) of system (3.10) is equiasymptotically x-stable.

4. EQUIASYMPTOTICAL STABILITY CRITERIA WITH TWO FUNCTIONS

THEOREM 4.1. *Let there exist continuous functionals* $V(t, \varphi)$ *and* $W(t, \varphi)$ *satisfying the following conditions*:

(i) *for every* $t_0 \in R_+$, *there exists* $\Delta = \Delta(t_0)$, *such that for each* $\varphi \in B_\Delta$ *there is a constant* $A = A(t_0, \varphi) > 0$ *for which*

$$-A \le V(t, z_t(t_0, \varphi)) \quad \textit{for all} \quad t \ge t_0; \tag{4.1}$$

(ii) $\frac{dV(t,z_t)}{dt} \le -W(t, z_t)$, $\quad W(t, 0) \equiv 0$, *and* $W(t, \varphi) \ge c(|(\psi(0)|)$, $c \in K$;

(iii) $\frac{dW(t,z_t)}{dt} \le 0$.

Then solution (2.1) *of system* (1.1) *is equiasymptotically x-stable*.

Proof. The functional $W(t, \varphi)$ is x-positive definite, and its derivative is nonpositive along solutions of system (1.1), so solution (2.1) is x-stable [18]. Hence, for every $t_0 \in R_+$ there exists $\delta = \delta(t_0)$ $(0 < \delta \le \Delta)$, such that $\varphi \in B_\delta$ implies $|x(t, t_0, \varphi)| < H$ for all $t \ge t_0$. Condition (iii) of the theorem implies that the function $W(t) = W(t, z_t(t_0, \varphi))$ does not increase. Show that

$$\lim_{t\to\infty} W(t) = 0, \tag{4.2}$$

if $\varphi \in B_\delta$. Assume the opposite: let $W(t) \ge l > 0$ for all $t \ge t_0$. Hence, $\dot{V}(t, z_t(t_0, \varphi)) \le -l$ for $t \ge t_0$, and from (4.1) we get

$$-A \le V(t, z_t(t_0, \varphi)) \le V(t_0, \varphi) - l(t - t_0),$$

which is impossible for sufficiently large t. This contradiction proves limit relation (4.2). From Theorem 2.1 it follows that solution (2.1) of system (1.1) is equiasymptotically x-stable.

In this particular case, when $\dot{V} = -W$, we have the following corollary.

COROLLARY 4.1. *If there exists a functional* $V(t, \varphi)$ *satisfying condition* (i) *of Theorem* 4.1 *and conditions*

(ii) $\dot{V}(t, 0) \equiv 0, \quad \dot{V}(t, \varphi) \leq -c(|\psi(0)|), \quad c \in K;$

(iii) $\ddot{V}(t, \varphi) \geq 0$, *then solution* (2.1) *of Eqs*. (1.1) *is equiasymptotically* x*-stable*.

Let $V(t, \varphi)$ and $W(t, \varphi)$ be continuous functionals on the set $R_+ \times C_H$. Suppose that V satisfies Lipschitz condition in t and ψ: $|V(t_1, \varphi) - V(t_2, \varphi)| \leq L|t_1 - t_2|$, $|V(t, \varphi_1) - V(t, \varphi_2)| \leq L\|\psi_1 - \psi_2\|$.

DEFINITION 4.1. A derivative $\frac{dW}{dt} = \frac{dW(t, z_t(t_0, \varphi))}{dt}$ satisfies condition (B) if there exists $q > 0$ ($q < H$), such that for any sufficiently small α ($\alpha < q$) there are a positive number $r = r(\alpha)$ and a continuous function $\xi_\alpha(t)$, that for any $t \in R_+$

$$\xi_\alpha(t) < 0, \qquad \int_t^{+\infty} \xi_\alpha(s)\, ds = -\infty, \tag{4.3}$$

and the inequality $dW/dt \leq \xi_\alpha(t)$ holds on G, where

$$G = \{\varphi \in C_H : \|\psi\| < q, V(t, \varphi) > \alpha, dV/dt > -r\}.$$

THEOREM 4.2. *Let the functional* $X(t, \varphi)$ *in system* (1.1) *be bounded in* $C_H(|X(t, \varphi)| < M)$. *If there exist continuous functionals* $V(t, \varphi)$ *and* $W(t, \varphi)$ *satisfying the following conditions*:

(1) $V(t, 0) \equiv 0,\ V(t, \varphi) \geq a(|\psi(0)|),\ a \in K;$

(2) $dV/dt \leq 0;$

(3) $|W(t, \varphi)| < N < +\infty;$

(4) dW/dt *satisfies condition* (*B*),

then solution (2.1) *of system* (1.1) *is equiasymptotically* x*-stable*.

Proof. A functional V is x-positive definite, so solution (2.1) of Eqs. (1.1) is x-stable. Let us show that it is equiasymptotically x-stable. Choose arbitrary positive q ($0 < q < H$).

For every $t_0 \in R_+$ there exists $\delta = \delta(t_0, q)$, such that for any $t \geq t_0$, $\varphi \in B_\delta$ the inequality $\|x_t(t_0, \varphi)\| < q$ is valid. Since $q \in (0, H)$ is fixed, then δ depends only on t_0; i.e., $\delta = \delta(t_0)$. Consider the trajectory $z(t_0, \varphi)$, where $\varphi \in B_\delta$. We choose δ in such way that $z_t(t_0, \varphi) \in C_q$ for all $t \geq t_0$.

From the condition (2) of the theorem, it follows that the function $V(t) = V(t, z_t(t_0, \varphi))$ is monotonically nonincreasing. Let us show that

$$\lim_{t\to\infty} V(t) = 0. \tag{4.4}$$

If (4.4) holds, then in view of Theorem 2.1 we derive that solution (2.1) is equiasymptotically x-stable.

Suppose that (4.4) is false; i.e., there exists $\alpha > 0$, such that

$$V(t) = V(t, z_t(t_0, \varphi)) \geq \alpha \quad \text{for } t \geq t_0. \tag{4.5}$$

Let us state some properties of the solution $z(t_0, \varphi)$.

Property 1. For any t_1 and t_2 the conditions $V(t_1, z_{t_1}(t_0, \varphi)) = r/2$, $V(t_2, z_{t_2}(t_0, \varphi) = r$ imply

$$|t_1 - t_2| \geq \frac{r}{2L(1+M)}. \tag{4.6}$$

In reality,

$$\begin{aligned} \frac{r}{2} &= |V(t_1, z_{t_1}) - V(t_2, z_{t_2})| \\ &\leq |V(t_1, z_{t_1}) - V(t_2, z_{t_1})| + |V(t_2, z_{t_1}) - V(t_2, z_{t_2})| \\ &\leq L|t_1 - t_2| + L\|x_{t_1} - x_{t_2}\|. \end{aligned} \tag{4.7}$$

By the finite increments formula, for any $i = 1, \ldots, n$ and t_1, t_2 we have

$$|x_i(t_2+\theta) - x_i(t_1+\theta)| = \left|\frac{dx_i(t_* + \theta)}{dt}\right| |t_2 - t_1| \leq M|t_2 - t_1|. \tag{4.8}$$

Combining (4.7) and (4.8), we get $r/2 \leq L(1+M)|t_1 - t_2|$. This inequality implies (4.6).

Property 2. The set G does not include z_t for all $t \geq t_0$.

Let $z_\tau \in G$. Assume that $z_t \in G$ for all $t > \tau$. For $t > \tau$ the inequalities

$$W(t, z_t) - W(\tau, z_\tau) \leq \int_\tau^t \dot{W}(s, z_s)\, ds \leq \int_\tau^t \xi_\alpha(s)\, ds \tag{4.9}$$

are valid. From relations (4.3) and (4.9) we get $\lim_{t\to+\infty} W(t, z_t) = -\infty$, which is in contradiction to the boundness of the functional $W(t, \varphi)$.

Property 3. If conditions $\|x_\tau\| < q$, $\dot{V}(\tau, z_\tau) \leq -r/2$ hold for $\tau \geq t_0$, then the inequality

$$V(\tau_1, z_{\tau_1}) < V(\tau, z_\tau) - \omega(\alpha), \tag{4.10}$$

where $\omega(\alpha) = \frac{r^2(\alpha)}{4L(M+1)}$, is valid for a moment τ_1, such that $\dot{V}(\tau_1, z_{\tau_1}) = -r$.

In fact, under the above conditions, there is a moment of time τ_2 ($\tau < \tau_2 < \tau_1$) such that $\dot{V}(\tau_2, z_{\tau_2}) = -r/2$, and for all $t \in [\tau_2, \tau_1]$ we have $-r \leq \dot{V}(t, z_t) \leq -r/2$. Properties 1 and 2 imply

$$\frac{r}{2L(M+1)} \leq \tau_1 - \tau_2,$$

whence it follows

$$\begin{aligned} V(\tau_1, z_{\tau_1}) - V(\tau, z_\tau) &\leq \int_\tau^{\tau_2} \dot{V}(s, z_s)\, ds + \int_{\tau_2}^{\tau_1} \dot{V}(s, z_s)\, ds < \int_{\tau_2}^{\tau_1} \dot{V}(s, z_s)\, ds \\ &< -\frac{r}{2}(\tau_1 - \tau_2) \leq -\frac{r^2(\alpha)}{4L(M+1)} \\ &= -\omega(\alpha). \end{aligned}$$

This completes the proof of Property 3.

Consider the sequence of moments of time $t_k = t_{k-1} + T_k, \quad k = 1, 2, \ldots$, where numbers $T_k = T_k(\alpha, t_0)$ are defined as follows

$$\int_{t_{k-1}}^{t_{k-1}+T_k^*} \xi_\alpha(s)\, ds = -(2N+1), \quad T_k = \max\left(T_k^*, \frac{r}{2L(M+1)}\right).$$

Property 4. The inequality

$$V(t_{k+2}, z_{t_{k+2}}) < V(t_k, z_{t_k}) - \omega \tag{4.11}$$

holds for every natural number k.

If for all $t \in [t_k, t_{k+1}]$ the inequality $\dot{V}(t, z_t) \leq -r/2$ holds, then

$$V(t_{k+2}, z_{t_{k+2}}) - V(t_k, z_{t_k}) \leq \int_{t_k}^{t_{k+1}} \dot{V}(s, z_s)\, ds \leq -\omega.$$

If there exists $\tau \in [t_k, t_{k+1}]$ such that $\dot{V}(\tau, z_\tau) > -r/2$, then there is such τ_* ($\tau < \tau_* < t_{k+2}$), that $\dot{V}(\tau_*, z_{\tau_*}) = -r$. According to Property 3, we have

$$V(t_{k+2}, z_{t_{k+2}}) \leq V(\tau_*, z_{\tau_*}) \leq V(\tau, z_\tau) - \omega \leq V(t_k, z_{t_k}) - \omega.$$

Property 4 is proved.

From (4.11) we obtain $V(t_{2k}, z_{2k}) \leq V(t_0, \varphi) - k\omega$. This inequality contradicts conjecture (1) for sufficiently large k. This completes the proof of the theorem.

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